

NONHOLONOMIC LAGRANGIAN SYSTEMS ON LIE ALGEBROIDS

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ABSTRACT. This paper presents a geometric description on Lie algebroids of Lagrangian systems subject to nonholonomic constraints. The Lie algebroid framework provides a natural generalization of classical tangent bundle geometry. We define the notion of nonholonomically constrained system, and characterize regularity conditions that guarantee that the dynamics of the system can be obtained as a suitable projection of the unconstrained dynamics. The proposed novel formalism provides new insights into the geometry of nonholonomic systems, and allows us to treat in a unified way a variety of situations, including systems with symmetry, morphisms, reduction, and nonlinearly constrained systems. Various examples illustrate the results.

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Date: April 30, 2008.

2000 Mathematics Subject Classification. 70F25, 70H03, 70H33, 37J60, 53D17.

Key words and phrases. Nonholonomic Mechanics, Lagrange-d'Alembert equations, Lie algebroids, symmetry, reduction.

This work has been partially supported by Spanish Ministry of Education and Culture grants MTM2004-7832, BFM2003-01319, MTM2006-03322 and BFM2003-02532. J. Cortés was partially supported by faculty research funds granted by the University of California, Santa Cruz.

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1. INTRODUCTION

The category of Lie algebroids has proved useful to formulate problems in applied mathematics, algebraic topology, and differential geometry. In the context of Mechanics, an ambitious program was proposed in [68] in order to develop formulations of the dynamical behavior of Lagrangian and Hamiltonian systems on Lie algebroids and discrete mechanics on Lie groupoids. In the last years, this program has been actively developed by many authors, and as a result, a powerful mathematical structure is emerging.

The main feature of the Lie algebroid framework is its inclusive nature. Under the same umbrella, one can consider such disparate situations as systems with symmetry, systems evolving on semidirect products, Lagrangian and Hamiltonian systems on Lie algebras, and field theory equations (see [19, 43] for recent topical reviews illustrating this). The Lie algebroid approach to Mechanics builds on the particular structure of the tangent bundle to develop a geometric treatment of Lagrangian systems parallel to Klein's formalism [22, 36]. At the same time, the attention devoted to Lie algebroids from a purely geometrical viewpoint has led to an spectacular development of the field, e.g., see [5, 13, 48, 60] and references therein. The merging of both perspectives has already provided mutual benefit, and will undoubtedly lead to important developments in the future.

The other main theme of this paper are nonholonomic Lagrangian systems, i.e., systems subject to constraints involving the velocities. This topic is a classic subject in Mathematics and Mechanics, dating back to the early times of Lagrange; a comprehensive list of classical references can be found in [57]. At the beginning of the nineties, the work [37] sparked a renewed interest in the geometric study of nonholonomic mechanical systems, with a special emphasis on symmetry aspects. In the last years, several authors have extended the ideas and techniques of the geometrical treatment of unconstrained systems to the study of nonholonomic mechanical systems, see the recent monographs [3, 17]. These include symplectic [12, 40, 41], Hamiltonian [65], and Lagrangian approaches [23, 38], the study of almost Poisson brackets [10, 33, 39], and symmetry and reduction of the dynamics [2, 4, 7, 8, 9, 18, 47].

In this paper we develop a comprehensive treatment of nonholonomic systems on Lie algebroids. This class of systems was introduced in [20] when studying mechanical control systems (see also [56] for a recent approach to mechanical systems on Lie algebroids subject to linear constraints). Here, we build on the geometry of Lie algebroids to identify suitable regularity conditions guaranteeing that the

nonholonomic system admits a unique solution. We develop a projection procedure to obtain the constrained dynamics as a modification of the unconstrained one, and define an almost-Poisson nonholonomic bracket. We show that many of the properties that standard nonholonomic systems enjoy have their counterpart in the proposed setup. As important examples, we highlight that the analysis here provides a natural interpretation for the use of pseudo-coordinates techniques and lends itself to the treatment of constrained systems with symmetry, following the ideas developed in [20, 53]. We carefully examine the reduction procedure for this class of systems, paying special attention to the evolution of the momentum map.

From a methodological point of view, the approach taken in the paper has enormous advantages. This fact must mainly be attributed to the inclusive nature of Lie algebroids. Usually, the results on nonholonomic systems available in the literature are restricted to a particular class of nonholonomic systems, or to a specific context. However, as illustrated in Table 1, many different nonholonomic systems fit under the Lie algebroid framework, and this has the important consequence of making the results proved here widely applicable. With the aim of illustrating this breadth, we consider various examples throughout the paper, including the Suslov problem, the Chaplygin sleigh, the Veselova system, Chaplygin Gyro-type systems, the two-wheeled planar mobile robot, and a ball rolling on a rotating table. We envision that future developments within the proposed framework will have a broad impact in nonholonomic mechanics. In the course of the preparation of this manuscript, the recent research efforts [14, 55] were brought to our attention. These references, similar in spirit to the present work, deal with nonholonomic Lagrangian systems and focus on the reduction of Lie algebroid structures under symmetry.

Nonholonomic Lagrangian system	Lie algebroid	Dynamics	Example
Standard	Tangent bundle	Lagrange-d'Alembert	Rolling disk [57]
On a Lie algebra	Lie algebra	Euler-Poincaré-Suslov	Chaplygin sleigh [16]
Nonholonomic LR systems	Right action Lie algebroid	Reduced Poincaré-Chetaev	Veselova problem [66]
Nonholonomic systems with semidirect product symmetry	Left action Lie algebroid	Nonholonomic Euler-Poincaré with an advected parameter	Chaplygin's gyro [49, 64]
Symmetry-invariant	Atiyah algebroid	Nonholonomic Lagrange-Poincaré	Snakeboard [4]

TABLE 1. The Lie algebroid framework embraces different classes of nonholonomic systems.

The paper is organized as follows. In Section 2 we collect some preliminary notions and geometric objects on Lie algebroids, including differential calculus, morphisms and prolongations. We also describe classical Lagrangian systems within the formalism of Lie algebroids. In Section 3, we introduce the class of nonholonomic Lagrangian systems subject to linear constraints, given by a regular Lagrangian $L : E \rightarrow \mathbb{R}$ on the Lie algebroid $\tau : E \rightarrow M$ and a constraint subbundle D of E . We show that the known results in Mechanics for these systems also hold in the context of Lie algebroids. In particular, drawing analogies with d'Alembert principle, we derive the Lagrange-d'Alembert equations of motion, prove the conservation of energy and state a Noether's theorem. We also derive local expressions for the dynamics of nonholonomic Lagrangian systems, which are further simplified by the choice of a convenient basis of D . As an illustration, we consider the class of nonholonomic mechanical systems. For such systems, the Lagrangian L is the polar

form of a bundle metric on E minus a potential function on M . In Section 3.2, we perform the analysis of the existence and uniqueness of solutions of constrained systems on general Lie algebroids, and extend the results in [2, 8, 10, 18, 40] for constrained systems evolving on tangent bundles. We obtain several characterizations for the regularity of a nonholonomic system, and prove that a nonholonomic system of mechanical type is always regular. The constrained dynamics can be obtained by projecting the unconstrained dynamics in two different ways. Under the first projection, we develop a distributional approach analogous to that in [2], see also [56]. Using the second projection, we introduce the nonholonomic bracket. The evolution of any observable can be measured by computing its bracket with the energy of the system. Section 4 is devoted to studying the reduction of the dynamics under symmetry. Our approach follows the ideas developed in [15], who defined a minimal subcategory of the category of Lie algebroids which is stable under Lagrangian reduction. We study the behavior of the different geometric objects introduced under morphisms of Lie algebroids, and show that fiberwise surjective morphisms induce consistent reductions of the dynamics. This result covers, but does not reduce to, the usual case of reduction of the dynamics by a symmetry group. In accordance with the philosophy of the paper, we study first the unconstrained dynamics case, and obtain later the results for the constrained dynamics using projections. A (Poisson) reduction by stages procedure can also be developed within this formalism. It should be noticed that the reduction under the presence of a Lie group of symmetries G is performed in two steps: first we reduce by a normal subgroup N of G , and then by the residual group. In Section 5, we prove a general version of the momentum equation introduced in [4]. In Section 6, we show some interesting examples and in Section 7, we extend some of the results previously obtained for linear constraints to the case of nonlinear constraints. The paper ends with our conclusions and a description of future research directions.

2. PRELIMINARIES

In this section we recall some well-known facts concerning the geometry of Lie algebroids. We refer the reader to [6, 32, 48] for details about Lie groupoids, Lie algebroids and their role in differential geometry.

2.1. Lie algebroids. Let M be an n -dimensional manifold and let $\tau: E \rightarrow M$ be a vector bundle. A vector bundle map $\rho: E \rightarrow TM$ over the identity is called an **anchor map**. The vector bundle E together with an anchor map ρ is said to be an **anchored vector bundle** (see [59]). A structure of **Lie algebroid** on E is given by a Lie algebra structure on the $C^\infty(M)$ -module of sections of the bundle, $(\text{Sec}(E), [\cdot, \cdot])$, together with an anchor map, satisfying the compatibility condition

$$[\sigma, f\eta] = f[\sigma, \eta] + (\rho(\sigma)f)\eta.$$

Here f is a smooth function on M , σ, η are sections of E and $\rho(\sigma)$ denotes the vector field on M given by $\rho(\sigma)(m) = \rho(\sigma(m))$. From the compatibility condition and the Jacobi identity, it follows that the map $\sigma \mapsto \rho(\sigma)$ is a Lie algebra homomorphism from the set of sections of E , $\text{Sec}(E)$, to the set of vector fields on M , $\mathfrak{X}(M)$.

In what concerns Mechanics, it is convenient to think of a Lie algebroid $\rho: E \rightarrow TM$, and more generally an anchored vector bundle, as a substitute of the tangent bundle of M . In this way, one regards an element a of E as a generalized velocity, and the actual velocity v is obtained when applying the anchor to a , i.e., $v = \rho(a)$. A curve $a: [t_0, t_1] \rightarrow E$ is said to be **admissible** if $\dot{m}(t) = \rho(a(t))$, where $m(t) = \tau(a(t))$ is the base curve. We will denote by $\text{Adm}(E)$ the space of admissible curves on E .

Given local coordinates (x^i) in the base manifold M and a local basis $\{e_\alpha\}$ of sections of E , we have local coordinates (x^i, y^α) in E . If $a \in E$ is an element in the fiber over $m \in M$, then we can write $a = y^\alpha e_\alpha(m)$ and thus the coordinates of a are (m^i, y^α) , where m^i are the coordinates of the point m . The anchor map is locally determined by the local functions ρ_α^i on M defined by $\rho(e_\alpha) = \rho_\alpha^i(\partial/\partial x^i)$. In addition, for a Lie algebroid, the Lie bracket is determined by the functions $C_{\alpha\beta}^\gamma$ defined by $[e_\alpha, e_\beta] = C_{\alpha\beta}^\gamma e_\gamma$. The functions ρ_α^i and $C_{\alpha\beta}^\gamma$ are called **the structure functions** of the Lie algebroid in this coordinate system. They satisfy the following relations

$$\rho_\alpha^j \frac{\partial \rho_\beta^i}{\partial x^j} - \rho_\beta^j \frac{\partial \rho_\alpha^i}{\partial x^j} = \rho_\gamma^i C_{\alpha\beta}^\gamma \quad \text{and} \quad \sum_{\text{cyclic}(\alpha, \beta, \gamma)} \left[\rho_\alpha^i \frac{\partial C_{\beta\gamma}^\nu}{\partial x^i} + C_{\beta\gamma}^\mu C_{\alpha\mu}^\nu \right] = 0,$$

which are called **the structure equations** of the Lie algebroid.

2.2. Exterior differential. The anchor ρ allows to define the differential of a function on the base manifold with respect to an element $a \in E$. It is given by

$$df(a) = \rho(a)f.$$

It follows that the differential of f at the point $m \in M$ is an element of E_m^* . Moreover, a structure of Lie algebroid on E allows to extend the differential to sections of the bundle $\bigwedge^p E$, which will be called p -sections or just p -forms. If $\omega \in \text{Sec}(\bigwedge^p E)$, then $d\omega \in \text{Sec}(\bigwedge^{p+1} E)$ is defined by

$$\begin{aligned} d\omega(\sigma_0, \sigma_1, \dots, \sigma_p) &= \sum_i (-1)^i \rho(\sigma_i)(\omega(\sigma_0, \dots, \widehat{\sigma}_i, \dots, \sigma_p)) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([\sigma_i, \sigma_j], \sigma_0, \dots, \widehat{\sigma}_i, \dots, \widehat{\sigma}_j, \dots, \sigma_p). \end{aligned}$$

It follows that d is a cohomology operator, that is, $d^2 = 0$. Locally the exterior differential is determined by

$$dx^i = \rho_\alpha^i e^\alpha \quad \text{and} \quad de^\gamma = -\frac{1}{2} C_{\alpha\beta}^\gamma e^\alpha \wedge e^\beta.$$

Throughout this paper, the symbol d will refer to the exterior differential on the Lie algebroid E and not to the ordinary exterior differential on a manifold. Of course, if $E = TM$, then both exterior differentials coincide.

The usual Cartan calculus extends to the case of Lie algebroids (see [48, 58]). For every section σ of E we have a derivation i_σ (contraction) of degree -1 and a derivation $d_\sigma = i_\sigma \circ d + d \circ i_\sigma$ (Lie derivative) of degree 0 . Since $d^2 = 0$, we have that $d_\sigma \circ d = d \circ d_\sigma$.

2.3. Morphisms. Let $\tau: E \rightarrow M$ and $\tau': E' \rightarrow M'$ be two anchored vector bundles, with anchor maps $\rho: E \rightarrow TM$ and $\rho': E' \rightarrow TM'$. A vector bundle map $\Phi: E \rightarrow E'$ over a map $\varphi: M \rightarrow M'$ is said to be **admissible** if it maps admissible curves onto admissible curves, or equivalently $T\varphi \circ \rho = \rho' \circ \Phi$. If E and E' are Lie algebroids, then we say that Φ is a **morphism** if $\Phi^* d\theta = d\Phi^* \theta$ for every $\theta \in \text{Sec}(\bigwedge^p E')$. It is easy to see that morphisms are admissible maps.

In the above expression, the pullback $\Phi^* \beta$ of a p -form β is defined by

$$(\Phi^* \beta)_m(a_1, a_2, \dots, a_p) = \beta_{\varphi(m)}(\Phi(a_1), \Phi(a_2), \dots, \Phi(a_p)),$$

for every $a_1, \dots, a_p \in E_m$. For a function $f \in C^\infty(M')$ (i.e., for $p = 0$), we just set $\Phi^* f = f \circ \varphi$.

Let (x^i) and (x'^i) be local coordinate systems on M and M' , respectively. Let $\{e_\alpha\}$ and $\{e'_\alpha\}$ be local basis of sections of E and E' , respectively, and $\{e^\alpha\}$ and

$\{e'^\alpha\}$ the corresponding dual basis. The bundle map Φ is determined by the relations $\Phi^*x'^i = \phi^i(x)$ and $\Phi^*e'^\alpha = \phi_\beta^\alpha e^\beta$ for certain local functions ϕ^i and ϕ_β^α on M . Then, Φ is admissible if and only if

$$\rho_\alpha^j \frac{\partial \phi^i}{\partial x^j} = \rho'^i_\beta \phi_\alpha^\beta.$$

The map Φ is a morphism of Lie algebroids if and only if, in addition to the admissibility condition above, one has

$$\phi_\gamma^\beta C_{\alpha\delta}^\gamma = \left(\rho_\alpha^i \frac{\partial \phi_\delta^\beta}{\partial x^i} - \rho_\delta^i \frac{\partial \phi_\alpha^\beta}{\partial x^i} \right) + C_{\theta\sigma}^{\prime\beta} \phi_\alpha^\theta \phi_\delta^\sigma.$$

In these expressions, ρ_α^i , $C_{\beta\gamma}^\alpha$ are the local structure functions on E and ρ'^i_α , $C_{\beta\gamma}^{\prime\alpha}$ are the local structure functions on E' .

2.4. Prolongation of a fibered manifold with respect to a Lie algebroid.

Let $\pi: P \rightarrow M$ be a fibered manifold with base manifold M . Thinking of E as a substitute of the tangent bundle of M , the tangent bundle of P is not the appropriate space to describe dynamical systems on P . This is clear if we note that the projection to M of a vector tangent to P is a vector tangent to M , and what one would like instead is an element of E , the ‘new’ tangent bundle of M .

A space which takes into account this restriction is the *E -tangent bundle* of P , also called the *prolongation* of P with respect to E , which we denote by $\mathcal{T}^E P$ (see [43, 51, 52, 59]). It is defined as the vector bundle $\tau_P^E: \mathcal{T}^E P \rightarrow P$ whose fiber at a point $p \in P_m$ is the vector space

$$\mathcal{T}_p^E P = \{ (b, v) \in E_m \times T_p P \mid \rho(b) = T_p \pi(v) \}.$$

We will frequently use the redundant notation (p, b, v) to denote the element $(b, v) \in \mathcal{T}_p^E P$. In this way, the map τ_P^E is just the projection onto the first factor. The anchor of $\mathcal{T}^E P$ is the projection onto the third factor, that is, the map $\rho^1: \mathcal{T}^E P \rightarrow TP$ given by $\rho^1(p, b, v) = v$. The projection onto the second factor will be denoted by $\mathcal{T}\pi: \mathcal{T}^E P \rightarrow E$, and it is a vector bundle map over π . Explicitly $\mathcal{T}\pi(p, b, v) = b$.

An element $z \in \mathcal{T}^E P$ is said to be vertical if it projects to zero, that is $\mathcal{T}\pi(z) = 0$. Therefore it is of the form $(p, 0, v)$, with v a vertical vector tangent to P at p .

Given local coordinates (x^i, u^A) on P and a local basis $\{e_\alpha\}$ of sections of E , we can define a local basis $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$ of sections of $\mathcal{T}^E P$ by

$$\mathcal{X}_\alpha(p) = \left(p, e_\alpha(\pi(p)), \rho_\alpha^i \frac{\partial}{\partial x^i} \Big|_p \right) \quad \text{and} \quad \mathcal{V}_A(p) = \left(p, 0, \frac{\partial}{\partial u^A} \Big|_p \right).$$

If $z = (p, b, v)$ is an element of $\mathcal{T}^E P$, with $b = z^\alpha e_\alpha$, then v is of the form $v = \rho_\alpha^i z^\alpha \frac{\partial}{\partial x^i} + v^A \frac{\partial}{\partial u^A}$, and we can write

$$z = z^\alpha \mathcal{X}_\alpha(p) + v^A \mathcal{V}_A(p).$$

Vertical elements are linear combinations of $\{\mathcal{V}_A\}$.

The anchor map ρ^1 applied to a section Z of $\mathcal{T}^E P$ with local expression $Z = Z^\alpha \mathcal{X}_\alpha + V^A \mathcal{V}_A$ is the vector field on P whose coordinate expression is

$$\rho^1(Z) = \rho_\alpha^i Z^\alpha \frac{\partial}{\partial x^i} + V^A \frac{\partial}{\partial u^A}.$$

If E carries a Lie algebroid structure, then so does $\mathcal{T}^E P$. The associated Lie bracket can be easily defined in terms of projectable sections, so that $\mathcal{T}\pi$ is a morphism of Lie algebroids. A section Z of $\mathcal{T}^E P$ is said to be projectable if there exists a section σ of E such that $\mathcal{T}\pi \circ Z = \sigma \circ \pi$. Equivalently, a section Z is projectable if and only if it is of the form $Z(p) = (p, \sigma(\pi(p)), X(p))$, for some

section σ of E and some vector field X on E (which projects to $\rho(\sigma)$). The Lie bracket of two projectable sections Z_1 and Z_2 is then given by

$$[Z_1, Z_2](p) = (p, [\sigma_1, \sigma_2](m), [X_1, X_2](p)), \quad p \in P, \quad m = \pi(p).$$

It is easy to see that $[Z_1, Z_2](p)$ is an element of $\mathcal{T}_p^E P$ for every $p \in P$. Since any section of $\mathcal{T}^E P$ can be locally written as a linear combination of projectable sections, the definition of the Lie bracket for arbitrary sections of $\mathcal{T}^E P$ follows.

The Lie brackets of the elements of the basis are

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = C_{\alpha\beta}^\gamma \mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{V}_B] = 0 \quad \text{and} \quad [\mathcal{V}_A, \mathcal{V}_B] = 0,$$

and the exterior differential is determined by

$$\begin{aligned} dx^i &= \rho_\alpha^i \mathcal{X}^\alpha, & du^A &= \mathcal{V}^A, \\ d\mathcal{X}^\gamma &= -\frac{1}{2} C_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, & d\mathcal{V}^A &= 0, \end{aligned}$$

where $\{\mathcal{X}^\alpha, \mathcal{V}^A\}$ is the dual basis corresponding to $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$.

2.5. Prolongation of a map. Let $\Psi: P \rightarrow P'$ be a fibered map from the fibered manifold $\pi: P \rightarrow M$ to the fibered manifold $\pi': P' \rightarrow M'$ over a map $\varphi: M \rightarrow M'$. Let $\Phi: E \rightarrow E'$ be an admissible map from $\tau: E \rightarrow M$ to $\tau': E' \rightarrow M'$ over the same map φ . The prolongation of Φ with respect to Ψ is the mapping $\mathcal{T}^\Phi \Psi: \mathcal{T}^E P \rightarrow \mathcal{T}^{E'} P'$ defined by

$$\mathcal{T}^\Phi \Psi(p, b, v) = (\Psi(p), \Phi(b), (T_p \Psi)(v)).$$

It is clear from the definition that $\mathcal{T}^\Phi \Psi$ is a vector bundle map from $\tau_P^E: \mathcal{T}^E P \rightarrow P$ to $\tau_{P'}^{E'}: \mathcal{T}^{E'} P' \rightarrow P'$ over Ψ . Moreover, in [54] it is proved the following result.

Proposition 2.1. *The map $\mathcal{T}^\Phi \Psi$ is an admissible map. Moreover, $\mathcal{T}^\Phi \Psi$ is a morphism of Lie algebroids if and only if Φ is a morphism of Lie algebroids.*

Given local coordinate systems (x^i) on M and (x'^i) on M' , local adapted coordinates (x^i, u^A) on P and (x'^i, u'^A) on P' and a local basis of sections $\{e_\alpha\}$ of E and $\{e'_\alpha\}$ of E' , the maps Φ and Ψ are determined by $\Phi^* e'^\alpha = \Phi_\beta^\alpha e^\beta$ and $\Psi(x, u) = (\phi^i(x), \psi^A(x, u))$. Then the action of $\mathcal{T}^\Phi \Psi$ is given by

$$\begin{aligned} (\mathcal{T}^\Phi \Psi)^* \mathcal{X}'^\alpha &= \Phi_\beta^\alpha \mathcal{X}^\beta, \\ (\mathcal{T}^\Phi \Psi)^* \mathcal{V}'^A &= \rho_\alpha^i \frac{\partial \psi^A}{\partial x^i} \mathcal{X}^\alpha + \frac{\partial \psi^A}{\partial u^B} \mathcal{V}^B. \end{aligned}$$

We finally mention that the composition of prolongation maps is the prolongation of the composition. Indeed, let Ψ' be another bundle map from $\pi': P' \rightarrow M'$ to another bundle $\pi'': P'' \rightarrow M''$ and Φ' be another admissible map from $\tau': E' \rightarrow M'$ to $\tau'': E'' \rightarrow M''$ both over the same base map. Since Φ and Φ' are admissible maps then so is $\Phi' \circ \Phi$, and thus we can define the prolongation of $\Psi' \circ \Psi$ with respect to $\Phi' \circ \Phi$. We have that $\mathcal{T}^{\Phi' \circ \Phi}(\Psi' \circ \Psi) = (\mathcal{T}^{\Phi'} \Psi') \circ (\mathcal{T}^\Phi \Psi)$.

In the particular case when the bundles P and P' are just $P = E$ and $P' = E'$, whenever we have an admissible map $\Phi: E \rightarrow E'$ we can define the prolongation of Φ along Φ itself, by $\mathcal{T}^\Phi \Phi(a, b, v) = (\Phi(a), \Phi(b), T\Phi(v))$. From the result above, we have that $\mathcal{T}^\Phi \Phi$ is a Lie algebroid morphism if and only if Φ is a Lie algebroid morphism. In coordinates we obtain

$$\begin{aligned} (\mathcal{T}^\Phi \Phi)^* \mathcal{X}'^\alpha &= \Phi_\beta^\alpha \mathcal{X}^\beta, \\ (\mathcal{T}^\Phi \Phi)^* \mathcal{V}'^\alpha &= \rho_\beta^i \frac{\partial \Phi_\gamma^\alpha}{\partial x^i} y^\gamma \mathcal{X}^\beta + \Phi_\beta^\alpha \mathcal{V}^\beta, \end{aligned}$$

where (x^i, y^γ) are the corresponding fibred coordinates on E . From this expression it is clear that $\mathcal{T}^\Phi \Phi$ is fiberwise surjective if and only if Φ is fiberwise surjective.

2.6. Lagrangian Mechanics. In [51] (see also [59]) a geometric formalism for Lagrangian Mechanics on Lie algebroids was defined. Such a formalism is similar to Klein's formalism [36] in standard Lagrangian mechanics and it is developed in the prolongation $\mathcal{T}^E E$ of a Lie algebroid E over itself. The canonical geometrical structures defined on $\mathcal{T}^E E$ are the following:

- The **vertical lift** $\xi^\vee : \tau^* E \rightarrow \mathcal{T}^E E$ given by $\xi^\vee(a, b) = (a, 0, b_a^\vee)$, where b_a^\vee is the vector tangent to the curve $a + tb$ at $t = 0$,
- The **vertical endomorphism** $S : \mathcal{T}^E E \rightarrow \mathcal{T}^E E$ defined as follows:

$$S(a, b, v) = \xi^\vee(a, b) = (a, 0, b_a^\vee),$$

- The **Liouville section** which is the vertical section corresponding to the Liouville dilation vector field:

$$\Delta(a) = \xi^\vee(a, a) = (a, 0, a_a^\vee).$$

A section Γ of $\mathcal{T}^E E$ is said to be a SODE section if $S\Gamma = \Delta$.

Given a Lagrangian function $L \in C^\infty(E)$ we define the **Cartan 1-form** θ_L and the **Cartan 2-form** ω_L as the forms on $\mathcal{T}^E E$ given by

$$\theta_L = S^*(dL) \quad \text{and} \quad \omega_L = -d\theta_L. \quad (2.1)$$

The real function E_L on E defined by $E_L = d_\Delta L - L$ is the **energy function** of the Lagrangian system.

By a solution of the Lagrangian system (a solution of the **Euler-Lagrange equations**) we mean a SODE section Γ of $\mathcal{T}^E E$ such that

$$i_\Gamma \omega_L - dE_L = 0. \quad (2.2)$$

The local expressions for the vertical endomorphism, the Liouville section, the Cartan 2-form and the Lagrangian energy are

$$S\mathcal{X}_\alpha = \mathcal{V}_\alpha, \quad S\mathcal{V}_\alpha = 0, \quad \text{for all } \alpha, \quad (2.3)$$

$$\Delta = y^\alpha \mathcal{V}_\alpha, \quad (2.4)$$

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \mathcal{X}^\alpha \wedge \mathcal{V}^\beta + \frac{1}{2} \left(\frac{\partial^2 L}{\partial x^i \partial y^\alpha} \rho_\beta^i - \frac{\partial^2 L}{\partial x^i \partial y^\beta} \rho_\alpha^i + \frac{\partial L}{\partial y^\gamma} C_{\alpha\beta}^\gamma \right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \quad (2.5)$$

$$E_L = \frac{\partial L}{\partial y^\alpha} y^\alpha - L. \quad (2.6)$$

Thus, a SODE Γ is a section of the form

$$\Gamma = y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha.$$

The SODE Γ is a solution of the Euler-Lagrange equations if and only if the functions f^α satisfy the linear equations

$$\frac{\partial^2 L}{\partial y^\beta \partial y^\alpha} f^\beta + \frac{\partial^2 L}{\partial x^i \partial y^\alpha} \rho_\beta^i y^\beta + \frac{\partial L}{\partial y^\gamma} C_{\alpha\beta}^\gamma y^\beta - \rho_\alpha^i \frac{\partial L}{\partial x^i} = 0, \quad \text{for all } \alpha. \quad (2.7)$$

The **Euler-Lagrange differential equations** are the differential equations for the integral curves of the vector field $\rho^1(\Gamma)$, where the section Γ is the solution of the Euler-Lagrange equations. Thus, these equations may be written as

$$\dot{x}^i = \rho_\alpha^i y^\alpha, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial y^\alpha} \right) - \rho_\alpha^i \frac{\partial L}{\partial x^i} + \frac{\partial L}{\partial y^\gamma} C_{\alpha\beta}^\gamma y^\beta = 0.$$

In other words, if $\delta L : \text{Adm}(E) \rightarrow E^*$ is the **Euler-Lagrange operator**, which locally reads

$$\delta L = \left(\frac{d}{dt} \left(\frac{\partial L}{\partial y^\alpha} \right) + C_{\alpha\beta}^\gamma y^\beta \frac{\partial L}{\partial y^\gamma} - \rho_\alpha^i \frac{\partial L}{\partial x^i} \right) e^\alpha,$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$, then the Euler-Lagrange differential equations read

$$\delta L = 0.$$

The function L is said to be **regular Lagrangian** if ω_L is regular at every point as a bilinear map. In such a case, there exists a unique section Γ_L of $\mathcal{T}^E E$ which satisfies the equation

$$i_{\Gamma_L} \omega_L - dE_L = 0.$$

Note that from (2.3), (2.4), (2.5) and (2.6), it follows that

$$i_{SX} \omega_L = -S^*(i_X \omega_L), \quad i_{\Delta} \omega_L = -S^*(dE_L), \quad (2.8)$$

for $X \in \text{Sec}(\mathcal{T}^E E)$. Thus, using (2.8), we deduce that

$$i_{S\Gamma_L} \omega_L = i_{\Delta} \omega_L$$

which implies that Γ_L is a SODE section. Therefore, for a regular Lagrangian function L we will say that the dynamical equations (2.2) are just the Euler-Lagrange equations.

On the other hand, the vertical distribution is isotropic with respect to ω_L , see [43]. This fact implies that the contraction of ω_L with a vertical vector is a semibasic form. This property allows us to define a symmetric 2-tensor G^L along τ by

$$G_a^L(b, c) = \omega_L(\tilde{b}, c_a^V), \quad (2.9)$$

where \tilde{b} is any element in $\mathcal{T}_a^E E$ which projects to b , i.e., $\mathcal{T}\tau(\tilde{b}) = b$, and $a \in E$. In coordinates $G^L = W_{\alpha\beta} e^\alpha \otimes e^\beta$, where the matrix $W_{\alpha\beta}$ is given by

$$W_{\alpha\beta} = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}. \quad (2.10)$$

It is easy to see that the Lagrangian L is regular if and only if the matrix W is regular at every point, that is, if the tensor G^L is regular at every point. By the kernel of G^L at a point a we mean the vector space

$$\text{Ker } G_a^L = \{ b \in E_{\tau(a)} \mid G_a^L(b, c) = 0 \text{ for all } c \in E_{\tau(a)} \}.$$

In the case of a regular Lagrangian, the Cartan 2-section ω_L is symplectic (non-degenerate and d -closed) and the vertical subbundle is Lagrangian. It follows that a 1-form is semi-basic if and only if it is the contraction of ω_L with a vertical element.

Finally, we mention that the **complete lift** σ^C of a section $\sigma \in \text{Sec}(E)$ is the section of $\mathcal{T}^E E$ characterized by the two following properties:

- (1) projects to σ , i.e., $\mathcal{T}\tau \circ \sigma^C = \sigma \circ \tau$,
- (2) $d_{\sigma^C} \hat{\mu} = \widehat{d_\sigma \mu}$,

where by $\hat{\alpha} \in C^\infty(E)$ we denote the linear function associated to a 1-section $\alpha \in \text{Sec}(E^*)$. Note that

$$\Gamma \text{ SODE section, } \sigma \in \text{Sec}(E) \Rightarrow S[\sigma^C, \Gamma] = 0, \quad (2.11)$$

$$S\gamma = 0, \quad \sigma \in \text{Sec}(E) \Rightarrow S[\sigma^C, \gamma] = 0. \quad (2.12)$$

3. LINEARLY CONSTRAINED LAGRANGIAN SYSTEMS

Nonholonomic systems on Lie algebroids were introduced in [20]. This class of systems includes, as particular cases, standard nonholonomic systems defined on the tangent bundle of a manifold and systems obtained by the reduction of the action of a symmetry group. The situation is similar to the non-constrained case, where the general equation $\delta L = 0$ comprises as particular cases the standard Lagrangian Mechanics, Lagrangian Mechanics with symmetry, Lagrangian systems with holonomic constraints, systems on semi-direct products and systems evolving on Lie algebras, see e.g., [51].

We start with a free Lagrangian system on a Lie algebroid E . As mentioned above, these two objects can describe a wide class of systems. Now, we plug in some nonholonomic linear constraints described by a subbundle D of the bundle E of admissible directions. If we impose to the solution curves $a(t)$ the condition to stay on the manifold D , we arrive at the equations $\delta L_{a(t)} = \lambda(t)$ and $a(t) \in D$, where the constraint force $\lambda(t) \in E_{\tau(a(t))}^*$ is to be determined. In the tangent bundle geometry case ($E = TM$), the d'Alembert principle establishes that the mechanical work done by the constraint forces vanishes, which implies that λ takes values in the annihilator of the constraint manifold D . Therefore, in the case of a general Lie algebroid, the natural equations one should pose are (see [20])

$$\delta L_{a(t)} \in D_{\tau(a(t))}^\circ \quad \text{and} \quad a(t) \in D.$$

In more explicit terms, we look for curves $a(t)$ on E such that

- they are admissible, $\rho(a(t)) = \dot{m}(t)$, where $m = \tau \circ a$,
- they stay in D , $a(t) \in D_{m(t)}$,
- there exists $\lambda(t) \in D_{m(t)}^\circ$ such that $\delta L_{a(t)} = \lambda(t)$.

If $a(t)$ is one of such curves, then $(a(t), \dot{a}(t))$ is a curve in $\mathcal{T}^E E$. Moreover, since $a(t)$ is in D , we have $\dot{a}(t)$ is tangent to D , that is, $(a(t), \dot{a}(t)) \in \mathcal{T}^D D$. Under some regularity conditions (to be made precise later on), we may assume that the above curves are integral curves of a section Γ , which as a consequence will be a SODE section taking values in $\mathcal{T}^D D$. Based on these arguments, we may reformulate geometrically our problem as the search for a SODE Γ (defined at least on a neighborhood of D) satisfying $(i_\Gamma \omega_L - dE_L)_a \in \widetilde{D_{\tau(a)}^\circ}$ and $\Gamma(a) \in \mathcal{T}_a^D D$, at every point $a \in D$. In the above expression $\widetilde{D^\circ}$ is the pullback of D° to $\mathcal{T}^E E$, that is, $\alpha \in \widetilde{D_{\tau(a)}^\circ}$ if and only if there exists $\lambda \in D_{\tau(a)}^\circ$ such that $\alpha = \lambda \circ \mathcal{T}_a \tau$.

Definition 3.1. A nonholonomically constrained Lagrangian system on a Lie algebroid E is a pair (L, D) , where L is a smooth function on E , the **Lagrangian**, and $i: D \hookrightarrow E$ is a smooth subbundle of E , known as the **constraint subbundle**. By a solution of the nonholonomically constrained Lagrangian system (L, D) we mean a SODE section $\Gamma \in \mathcal{T}^E E$ which satisfies the **Lagrange-d'Alembert equations**

$$\begin{aligned} (i_\Gamma \omega_L - dE_L)|_D &\in \text{Sec}(\widetilde{D^\circ}), \\ \Gamma|_D &\in \text{Sec}(\mathcal{T}^D D). \end{aligned} \tag{3.1}$$

With a slight abuse of language, we will interchangeably refer to a solution of the constrained Lagrangian system as a section or the collection of its corresponding integral curves. The restriction of the projection $\tau: E \rightarrow M$ to D will be denoted by π , that is, $\pi = \tau|_D: D \rightarrow M$.

Remark 3.2 (Domain of definition of solutions of the Lagrange-d'Alembert equations). We want to stress that a solution of the Lagrange-d'Alembert equations needs to be defined only over D , but for practical purposes we consider it extended to E (or just to a neighborhood of D in E). We will not make any notational distinction between a solution on D and any of its extensions. Solutions which coincide on D will be considered as equal. See [30, 40] for a more in-depth discussion. In accordance with this convention, by a SODE on D we mean a section of $\mathcal{T}^D D$ which is the restriction to D of some SODE defined in a neighborhood of D . Alternatively, a SODE on D is a section Γ of $\mathcal{T}^D D$ such that $\mathcal{T}\tau(\Gamma(a)) = a$ for every $a \in D$. •

Remark 3.3 (Holonomic constraints). A nonholonomically constrained Lagrangian system (L, D) on a Lie algebroid E is said to be **holonomic** if D is a Lie subalgebroid of E . This means that $[X, Y] \in \text{Sec}(D)$, for $X, Y \in \text{Sec}(D)$. Thus, the real

function $L_D = L|_D : D \rightarrow \mathbb{R}$ defines an unconstrained (free) Lagrangian system on the Lie algebroid D . Moreover, it is easy to prove that $\mathcal{I} \circ \Delta_D = \Delta \circ i$ and $\mathcal{I} \circ S_D = S \circ \mathcal{I}$, where $\mathcal{I} = T^i i : T^D D \rightarrow T^E E$ is the prolongation of the Lie algebroid morphism $i : D \hookrightarrow E$ and Δ_D (respectively, S_D) is the Liouville section (respectively, the vertical endomorphism) of the Lie algebroid $T^D D$. Therefore, since $L \circ i = L_D$, we deduce that

$$\mathcal{I}^*(\theta_L) = \theta_{L_D}, \quad \mathcal{I}^*(\omega_L) = \omega_{L_D}, \quad \mathcal{I}^*(dE_L) = dE_{L_D}.$$

Consequently, if Γ is a SODE section of $T^E E$, $a, b \in D$, $(b, X) \in \mathcal{T}_a^E E$ and $(b, Y) \in \mathcal{T}_a^E D$ then

$$(i_\Gamma \omega_L - dE_L)(a)(b, X) = (i_{\Gamma|_D} \omega_{L_D} - dE_{L_D})(a)(b, Y) + (i_\Gamma \omega_L - dE_L)(a)(0, Z),$$

$(0, Z)$ being a vertical element of $\mathcal{T}_a^E E$.

Now, using (2.8), we have that $(i_\Gamma \omega_L - dE_L)(a)(0, Z) = 0$ which implies that

$$(i_\Gamma \omega_L - dE_L)(a)(b, X) = (i_{\Gamma|_D} \omega_{L_D} - dE_{L_D})(a)(b, Y).$$

The above facts prove that a SODE section Γ of $T^E E$ is a solution of the holonomic Lagrangian system (L, D) on E if and only if $\Gamma|_D$ is a solution of the Euler-Lagrange equations for the (unconstrained) Lagrangian function L_D on the Lie algebroid D . In other words, the holonomic Lagrangian system (L, D) on E may be considered as an unconstrained (free) Lagrangian system on the Lie algebroid D . \bullet

Next, suppose that (L, D) is a nonholonomically constrained Lagrangian system on the Lie algebroid E . Then, the different spaces we will consider are shown in the following commutative diagram

$$\begin{array}{ccccc}
 & & TM & \xlongequal{\quad} & TM \\
 & & \uparrow \rho_D & & \uparrow \rho \\
 & & D & \xrightarrow{i} & E \\
 & & \uparrow T\pi & & \uparrow T\tau \\
 TD & \xleftarrow{\rho^1} & T^D D & \xrightarrow{\mathcal{I}} & T^E E & \xrightarrow{\rho^1} & TE \\
 & \searrow \tau_D & \downarrow \pi_D^D & & \downarrow \tau_E^E & \searrow \tau_E \\
 & & D & \xrightarrow{i} & E \\
 & & \downarrow \pi & & \downarrow \tau \\
 & & M & \xlongequal{\quad} & M
 \end{array}$$

As an intermediate space in our analysis of the regularity of the constrained systems, we will also consider $T^E D$, the E -tangent to D . The main difference between $T^E D$ and $T^D D$ is that the former has a natural Lie algebroid structure while the later does not.

The following two results are immediate consequences of the above form of the Lagrange-d'Alembert equations.

Theorem 3.4 (Conservation of energy). *If (L, D) is a constrained Lagrangian system and Γ is a solution of the dynamics, then $d_\Gamma E_L = 0$ (on D).*

Proof. Indeed, for every $a \in D$, we have $\Gamma(a) \in \mathcal{T}_a^D D$, so that $T\tau(\Gamma(a)) \in D$. Therefore $i_\Gamma \widetilde{D}^\circ = 0$ and contracting $0 = i_\Gamma(i_\Gamma \omega_L - dE_L) = -d_\Gamma E_L$ at every point in D . \square

Theorem 3.5 (Noether's theorem). *Let (L, D) be a constrained Lagrangian system which admits a unique SODE Γ solution of the dynamics. If σ is a section of D such that there exists a function $f \in C^\infty(M)$ satisfying*

$$d_{\sigma^c} L = \dot{f},$$

then the function $F = \langle \theta_L, \sigma^c \rangle - f$ is a constant of the motion, that is, $d_\Gamma F = 0$ (on D).

Proof. Using that $\theta_L(\Gamma) = d_\Delta(L)$, we obtain $i_{\sigma^c}(i_\Gamma \omega_L - dE_L) = i_{\sigma^c}(-d_\Gamma \theta_L + dL) = d_{\sigma^c} L - d_\Gamma \langle \theta_L, \sigma^c \rangle + \theta_L[\Gamma, \sigma^c]$ and, since $[\Gamma, \sigma^c]$ is vertical, we deduce

$$i_{\sigma^c}(i_\Gamma \omega_L - dE_L) = d_{\sigma^c} L - d_\Gamma \langle \theta_L, \sigma^c \rangle.$$

Thus, taking into account that $i_{\sigma^c} \widetilde{D}^\circ = 0$, we get $0 = d_\Gamma(\langle \theta_L, \sigma^c \rangle - f) = -d_\Gamma F$. \square

Example 3.6 (Mechanical systems with nonholonomic constraints). Let $\mathcal{G} : E \times_M E \rightarrow \mathbb{R}$ be a bundle metric on E . The **Levi-Civita connection** $\nabla^\mathcal{G}$ is determined by the formula

$$\begin{aligned} 2\mathcal{G}(\nabla_\sigma^\mathcal{G} \eta, \zeta) &= \rho(\sigma)(\mathcal{G}(\eta, \zeta)) + \rho(\eta)(\mathcal{G}(\sigma, \zeta)) - \rho(\zeta)(\mathcal{G}(\eta, \sigma)) \\ &\quad + \mathcal{G}(\sigma, [\zeta, \eta]) + \mathcal{G}(\eta, [\zeta, \sigma]) - \mathcal{G}(\zeta, [\eta, \sigma]), \end{aligned}$$

for $\sigma, \eta, \zeta \in \text{Sec}(E)$. The coefficients of the connection $\nabla^\mathcal{G}$ are given by

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} \mathcal{G}^{\alpha\nu} ([\nu, \beta; \gamma] + [\nu, \gamma; \beta] + [\beta, \gamma; \nu]),$$

where $\mathcal{G}_{\alpha\nu}$ are the coefficients of the metric \mathcal{G} , $(\mathcal{G}^{\alpha\nu})$ is the inverse matrix of $(\mathcal{G}_{\alpha\nu})$ and

$$[\alpha, \beta; \gamma] = \frac{\partial \mathcal{G}_{\alpha\beta}}{\partial x^i} \rho_\gamma^i + C_{\alpha\beta}^\mu \mathcal{G}_{\mu\gamma}.$$

Using the covariant derivative induced by $\nabla^\mathcal{G}$, one may introduce the notion of a geodesic of $\nabla^\mathcal{G}$ as follows. An admissible curve $a : I \rightarrow E$ is said to be a **geodesic** if $\nabla_{a(t)}^\mathcal{G} a(t) = 0$, for all $t \in I$. In local coordinates, the conditions for being a geodesic read

$$\frac{da^\gamma}{dt} + \frac{1}{2}(\Gamma_{\alpha\beta}^\gamma + \Gamma_{\beta\alpha}^\gamma) a^\alpha a^\beta = 0, \quad \text{for all } \gamma.$$

The geodesics are the integral curves of a SODE section $\Gamma_{\nabla^\mathcal{G}}$ of $\mathcal{T}^E E$, which is locally given by

$$\Gamma_{\nabla^\mathcal{G}} = y^\gamma \mathcal{X}_\gamma - \frac{1}{2}(\Gamma_{\alpha\beta}^\gamma + \Gamma_{\beta\alpha}^\gamma) y^\alpha y^\beta \mathcal{V}_\gamma.$$

$\Gamma_{\nabla^\mathcal{G}}$ is called the **geodesic flow** (for more details, see [20]).

The class of systems that were considered in detail in [20] is that of **mechanical systems with nonholonomic constraints**¹. The Lagrangian function L is of mechanical type, i.e., it is of the form

$$L(a) = \frac{1}{2} \mathcal{G}(a, a) - V(\tau(a)), \quad a \in E,$$

with V a function on M .

The Euler-Lagrange section for the unconstrained system can be written as

$$\Gamma_L = \Gamma_{\nabla^\mathcal{G}} - (\text{grad}_\mathcal{G} V)^\vee.$$

In this expression, by $\text{grad}_\mathcal{G} V$ we mean the section of E such that $\langle dV(m), a \rangle = \mathcal{G}(\text{grad}_\mathcal{G} V(m), a)$, for all $m \in M$ and all $a \in E_m$, and where we remind that d is

¹In fact, in [20], we considered controlled mechanical systems with nonholonomic constraints, that is, mechanical systems evolving on Lie algebroids and subject to some external control forces.

the differential in the Lie algebroid. The Euler-Lagrange differential equations can be written as

$$\begin{aligned}\dot{x}^i &= \rho_\alpha^i y^\alpha, \\ \dot{y}^\alpha &= -\frac{1}{2} (\Gamma_{\beta\gamma}^\alpha + \Gamma_{\gamma\beta}^\alpha) y^\beta y^\gamma - \mathcal{G}^{\alpha\beta} \rho_\beta^i \frac{\partial V}{\partial x^i}.\end{aligned}\tag{3.2}$$

Alternatively, one can describe the dynamical behavior of the mechanical control system by means of an equation on E via the covariant derivative. An admissible curve $a : t \mapsto a(t)$ with base curve $t \mapsto m(t)$ is a solution of the system (3.2) if and only if

$$\nabla_{a(t)}^{\mathcal{G}} a(t) + \text{grad}_{\mathcal{G}} V(m(t)) = 0.\tag{3.3}$$

Note that

$$\mathcal{G}(m(t))(\nabla_{a(t)}^{\mathcal{G}} a(t) + \text{grad}_{\mathcal{G}} V(m(t)), b) = \delta L(a(t))(b), \quad \text{for } b \in E_{m(t)}.$$

If this mechanical control system is subject to the constraints determined by a subbundle D of E , we can do the following. Consider the orthogonal decomposition $E = D \oplus D^\perp$, and the associated orthogonal projectors $P : E \rightarrow D$, $Q : E \rightarrow D^\perp$. Using the fact that $\mathcal{G}(P \cdot, \cdot) = \mathcal{G}(\cdot, P \cdot)$, one can write the Lagrange-d'Alembert equations in the form

$$P(\nabla_{a(t)}^{\mathcal{G}} a(t)) + P(\text{grad}_{\mathcal{G}} V(m(t))) = 0, \quad Q(a) = 0.$$

A specially well-suited form of these equations makes use of the **constrained connection** $\check{\nabla}$ defined by $\check{\nabla}_\sigma \eta = P(\nabla_\sigma^{\mathcal{G}} \eta) + \nabla_\sigma^{\mathcal{G}}(Q\eta)$. In terms of $\check{\nabla}$, we can rewrite this equation as $\check{\nabla}_{a(t)} a(t) + P(\text{grad}_{\mathcal{G}} V(m(t))) = 0$, $Q(a) = 0$, where we have used the fact that the connection $\check{\nabla}$ restricts to the subbundle D .

Moreover, following the ideas in [45], we proved in [20] that the subbundle D is geodesically invariant for the connection $\check{\nabla}$, that is, any integral curve of the spray $\Gamma_{\check{\nabla}}$ associated with $\check{\nabla}$ starting from a point in D is entirely contained in D . Since the terms coming from the potential V also belongs to D , we have that the constrained equations of motion can be simply stated as

$$\check{\nabla}_{a(t)} a(t) + P(\text{grad}_{\mathcal{G}} V(m(t))) = 0, \quad a(0) \in D.\tag{3.4}$$

Note that one can write the constrained equations of the motion as follows

$$\dot{a}(t) = \rho^1(\Gamma_{\check{\nabla}}(a(t)) - P(\text{grad}_{\mathcal{G}} V)^v(a(t)))$$

and that the restriction to D of the vector field $\rho^1(\Gamma_{\check{\nabla}} - P(\text{grad}_{\mathcal{G}} V)^v)$ is tangent to D .

The coordinate expression of equations (3.4) is greatly simplified if we take a basis $\{e_\alpha\} = \{e_a, e_A\}$ of E adapted to the orthogonal decomposition $E = D \oplus D^\perp$, i.e., $D = \text{span}\{e_a\}$, $D^\perp = \text{span}\{e_A\}$. Denoting by $(y^\alpha) = (y^a, y^A)$ the induced coordinates, the constraint equations $Q(a) = 0$ just read $y^A = 0$. The differential equations of the motion are then

$$\begin{aligned}\dot{x}^i &= \rho_a^i y^a, \\ \dot{y}^a &= -\frac{1}{2} (\check{\Gamma}_{bc}^a + \check{\Gamma}_{cb}^a) y^b y^c - \mathcal{G}^{ab} \rho_b^i \frac{\partial V}{\partial x^i}, \\ y^A &= 0,\end{aligned}$$

where $\check{\Gamma}_{\beta\gamma}^\alpha$ are the connection coefficients of the constrained connection $\check{\nabla}$. •

In the above example the dynamics exists and is completely determined whatever the (linear) constraints are. As we will see in Section 3.2, this property is lost in the general case.

3.1. Lagrange-d'Alembert equations in local coordinates. Let us analyze the form of the Lagrange-d'Alembert equations in local coordinates. Following the example above, let us choose a special coordinate system adapted to the structure of the problem as follows. We consider local coordinates (x^i) on an open set \mathcal{U} of M and we take a basis $\{e_a\}$ of local sections of D and complete it to a basis $\{e_a, e_A\}$ of local sections of E (both defined on the open \mathcal{U}). In this way, we have coordinates (x^i, y^a, y^A) on E . In this set of coordinates, the constraints imposed by the submanifold $D \subset E$ are simply $y^A = 0$. If $\{e^a, e^A\}$ is the dual basis of $\{e_a, e_A\}$, then a basis for the annihilator D° of D is $\{e^A\}$ and a basis for \widetilde{D}° is \mathcal{X}^A .

An element z of $T^E D$ is of the form $z = u^\alpha \mathcal{X}_\alpha + z^a \mathcal{V}_a = u^a \mathcal{X}_a + u^A \mathcal{X}_A + z^a \mathcal{V}_a$, that is, the component \mathcal{V}_A vanishes since $\rho^1(z)$ is a vector tangent to the manifold D with equations $y^A = 0$. The projection of z to E is $\mathcal{T}\tau(z) = u^a e_a + u^A e_A$, so that the element z is in $\mathcal{T}^D D$ if and only if $u^A = 0$. In other words, an element in $\mathcal{T}^D D$ is of the form $z = u^a \mathcal{X}_a + z^a \mathcal{V}_a$.

Let us find the local expression of the Lagrange-d'Alembert equations in these coordinates. We consider a section Γ such that $\Gamma|_D \in \text{Sec}(\mathcal{T}^D D)$, which is therefore of the form $\Gamma = g^a \mathcal{X}_a + f^a \mathcal{V}_a$. From the local expression (2.5) of the Cartan 2-form and the local expression (2.6) of the energy function, we get

$$0 = \langle i_\Gamma \omega_L - dE_L, \mathcal{V}_\alpha \rangle = -y^B \frac{\partial^2 L}{\partial y^\alpha \partial y^B} - (y^b - g^b) \frac{\partial^2 L}{\partial y^\alpha \partial y^b}.$$

If we assume that the Lagrangian L is regular, when we evaluate at $y^A = 0$, we have that $g^a = y^a$ and thus Γ is a SODE. Moreover, contracting with \mathcal{X}_a , after a few calculations we get

$$0 = \langle i_\Gamma \omega_L - dE_L, \mathcal{X}_a \rangle = - \left\{ d_\Gamma \left(\frac{\partial L}{\partial y^a} \right) + \frac{\partial L}{\partial y^\gamma} C_{a\beta}^\gamma y^\beta - \rho_a^i \frac{\partial L}{\partial x^i} \right\},$$

so that (again after evaluation at $y^A = 0$), the functions f^a are solution of the linear equations

$$\frac{\partial^2 L}{\partial y^b \partial y^a} f^b + \frac{\partial^2 L}{\partial x^i \partial y^a} \rho_b^i y^b + \frac{\partial L}{\partial y^\gamma} C_{ab}^\gamma y^b - \rho_a^i \frac{\partial L}{\partial x^i} = 0, \quad (3.5)$$

where all the partial derivatives of the Lagrangian are to be evaluated on $y^A = 0$.

As a consequence, we get that there exists a unique solution of the Lagrange-d'Alembert equations if and only if the matrix

$$\mathcal{C}_{ab}(x^i, y^c) = \frac{\partial^2 L}{\partial y^a \partial y^b}(x^i, y^c, 0) \quad (3.6)$$

is regular. Notice that \mathcal{C}_{ab} is a submatrix of $W_{\alpha\beta}$, evaluated at $y^A = 0$ and that, as we know, if L is of mechanical type then the Lagrange-d'Alembert equations have a unique solution. The differential equations for the integral curves of the vector field $\rho^1(\Gamma)$ are the Lagrange-d'Alembert differential equations, which read

$$\begin{aligned} \dot{x}^i &= \rho_a^i y^a, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial y^a} \right) + \frac{\partial L}{\partial y^\gamma} C_{ab}^\gamma y^b - \rho_a^i \frac{\partial L}{\partial x^i} &= 0, \\ y^A &= 0. \end{aligned} \quad (3.7)$$

Finally, notice that the contraction with \mathcal{X}_A just gives the components $\lambda_A = \langle i_\Gamma \omega_L - dE_L, \mathcal{X}_A \rangle|_{y^A=0}$ of the constraint forces $\lambda = \lambda_A e^A$.

Remark 3.7 (Equations in terms of the constrained Lagrangian). In some occasions, it is useful to write the equations in the form

$$\begin{aligned} \dot{x}^i &= \rho_a^i y^a, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial y^a} \right) + \frac{\partial L}{\partial y^c} C_{ab}^c y^b - \rho_a^i \frac{\partial L}{\partial x^i} &= - \frac{\partial L}{\partial y^A} C_{ab}^A y^b, \\ y^A &= 0, \end{aligned} \quad (3.8)$$

where, on the left-hand side of the second equation, all the derivatives can be calculated from the value of the Lagrangian on the constraint submanifold D . In other words, we can substitute L by the constrained Lagrangian L_c defined by $L_c(x^i, y^a) = L(x^i, y^a, 0)$. •

Remark 3.8 (Lagrange-d'Alembert equations in quasicordinates). A particular case of this construction is given by constrained systems defined in the standard Lie algebroid $\tau_M: TM \rightarrow M$. In this case, the equations (3.7) are the Lagrange-d'Alembert equations written in quasicordinates, where $C_{\beta\gamma}^\alpha$ are the so-called Hamel's transpositional symbols, which obviously are nothing but the structure coefficients (in the Cartan's sense) of the moving frame $\{e_\alpha\}$, see e.g., [25, 31]. •

3.2. Solution of Lagrange-d'Alembert equations.

Assumption 3.9. *In what follows, we will assume that the Lagrangian L is regular at least in a neighborhood of D .*

Let us now perform a precise global analysis of the existence and uniqueness of the solution of Lagrange-d'Alembert equations.

Definition 3.10. *A constrained Lagrangian system (L, D) is said to be **regular** if the Lagrange-d'Alembert equations have a unique solution.*

In order to characterize geometrically those nonholonomic systems which are regular, we define the tensor $G^{L,D}$ as the restriction of G^L to D , that is, $G_a^{L,D}(b, c) = G_a^L(b, c)$ for every $a \in D$ and every $b, c \in D_{\tau(a)}$. In coordinates adapted to D , we have that the local expression of $G^{L,D}$ is $G^{L,D} = C_{ab} e^a \otimes e^b$ where the matrix C_{ab} is given by equation (3.6).

A second important geometric object is the subbundle $F \subset \mathcal{T}^E E|_D \rightarrow D$ whose fiber at the point $a \in D$ is $F_a = \omega_L^{-1}(\widetilde{D_{\tau(a)}^\circ})$. More explicitly,

$$F_a = \{z \in \mathcal{T}_a^E E \mid \text{exists } \zeta \in D_{\tau(a)}^\circ \text{ s.t. } \omega_L(z, u) = \langle \zeta, \mathcal{T}\tau(u) \rangle \text{ for all } u \in \mathcal{T}_a^E E\}.$$

From the definition, it is clear that the rank of F is $\text{rank}(F) = \text{rank}(D^\circ) = \text{rank}(E) - \text{rank}(D)$.

Finally, we also consider the subbundle $(\mathcal{T}^D D)^\perp \subset \mathcal{T}^E E|_D \rightarrow D$, the orthogonal to $\mathcal{T}^D D$ with respect to the symplectic form ω_L . The rank of $(\mathcal{T}^D D)^\perp$ is $\text{rank}(\mathcal{T}^D D)^\perp = \text{rank}(\mathcal{T}^E E) - \text{rank}(\mathcal{T}^D D) = 2(\text{rank}(E) - \text{rank}(D)) = 2\text{rank}(D^\circ)$.

The relation among these three objects is described by the following result.

Lemma 3.11. *The following properties are satisfied:*

- (1) *The elements in F are vertical. An element $\xi^V(a, b) \in F_a$ if and only if $G_a^L(b, c) = 0$ for all $c \in D_{\tau(a)}$.*
- (2) *$(\mathcal{T}^D D)^\perp \cap \text{Ver}(\mathcal{T}^E E) = F$.*

Proof. (1) The elements in F are vertical because the elements in $\widetilde{D^\circ}$ are semi-basic. If $\xi^V(a, b) \in F_a$ then there exists $\zeta \in D_{\tau(a)}^\circ$ such that $\omega_L(\xi^V(a, b), u) = \langle \zeta, \mathcal{T}\tau(u) \rangle$ for all $u \in \mathcal{T}_a^E E$. In terms of G^L and writing $c = \mathcal{T}\tau(u)$, the above equation reads $-G_a^L(b, c) = \langle \zeta, c \rangle$. By taking $u \in \mathcal{T}\tau^{-1}(D)$, then c is in D and therefore

$G_a^L(b, c) = 0$ for all $c \in D_{\tau(a)}$. Conversely, if $G_a^L(b, c) = 0$ for all $c \in D_{\tau(a)}$, then the 1-form $\zeta = -G_a^L(b, \cdot)$ is in $D_{\tau(a)}^\circ$. Therefore $\omega_L(\xi^V(a, b), u) = -G_a^L(b, T\tau(u)) = \langle \zeta, T\tau(u) \rangle$, which is the condition for $\xi^V(a, b) \in F_a$.

(2) The condition for a vertical element $\xi^V(a, b)$ to be in $(\mathcal{T}^D D)^\perp$ is $\omega_L(\xi^V(a, b), w) = 0$ for all $w \in \mathcal{T}_a^D D$, or equivalently, $G_a^L(b, T\tau(w)) = 0$. The vector $c = T\tau(w)$ is an arbitrary element of $D_{\tau(a)}$, so that the above condition reads $G_a^L(b, c) = 0$, for all $c \in D_{\tau(a)}$, which is precisely the condition for $\xi^V(a, b)$ to be in F_a . \square

Theorem 3.12. *The following properties are equivalent:*

- (1) *The constrained Lagrangian system (L, D) is regular,*
- (2) $\text{Ker } G^{L,D} = \{0\},$
- (3) $\mathcal{T}^E D \cap F = \{0\},$
- (4) $\mathcal{T}^D D \cap (\mathcal{T}^D D)^\perp = \{0\}.$

Proof. [(1) \Leftrightarrow (2)] The equivalence between the first two conditions is clear from the local form of the Lagrange-d'Alembert equations (3.5), since the coefficients of the unknowns f^a are precisely the components (3.6) of $G^{L,D}$.

[(2) \Leftrightarrow (3)] (\Rightarrow) Let $a \in D$ and consider an element $z \in \mathcal{T}_a^E D \cap F_a$. Since the elements of F are vertical, we have $z = \xi^V(a, b)$ for some $b \in E_{\tau(a)}$. Moreover, $z \in \mathcal{T}_a^E D$ implies that b is an element in $D_{\tau(a)}$. On the other hand, if $z = \xi^V(a, b)$ is in F_a , then Lemma 3.11 implies that $G_a^L(b, c) = 0$ for all $c \in D_{\tau(a)}$. Thus $G_a^{L,D}(b, c) = 0$ for all $c \in D_{\tau(a)}$, from where $b = 0$, and hence $z = 0$.

(\Leftarrow) Conversely, if for some $a \in D$, there exists $b \in \text{Ker } G_a^{L,D}$ with $b \neq 0$ then, using Lemma 3.11, we deduce that $z = \xi^V(a, b) \in \mathcal{T}_a^E D \cap F_a$ and $z \neq 0$.

[(2) \Leftrightarrow (4)] (\Rightarrow) Let $a \in D$ and consider an element $v \in \mathcal{T}_a^D D \cap (\mathcal{T}_a^D D)^\perp$, that is, $\omega_L(v, w) = 0$ for all $w \in \mathcal{T}_a^D D$. If we take $w = \xi^V(a, b)$ for $b \in D_{\tau(a)}$ arbitrary, then we have $\omega_L(v, \xi^V(a, b)) = G_a^{L,D}(T\tau(v), b) = 0$ for all $b \in D_{\tau(a)}$, from where it follows that $T\tau(v) = 0$. Thus v is vertical, $v = \xi^V(a, c)$, for some $c \in D$ and then $\omega_L(\xi^V(a, c), w) = -G_a^{L,D}(c, T\tau(w)) = 0$ for all $w \in \mathcal{T}_a^D D$. Therefore $c = 0$ and hence $v = 0$.

(\Leftarrow) Conversely, if for some $a \in D$, there exists $b \in \text{Ker } G_a^{L,D}$ with $b \neq 0$, then $0 \neq \xi^V(a, b) \in \mathcal{T}_a^D D \cap (\mathcal{T}_a^D D)^\perp$, because $\omega_L(\xi^V(a, b), w) = G_a^{L,D}(b, T\tau(w)) = 0$ for all $w \in \mathcal{T}_a^D D$. \square

In the case of a constrained mechanical system, the tensor G^L is given by $G_a^L(b, c) = \mathcal{G}_{\tau(a)}(b, c)$, so that it is positive definite at every point. Thus the restriction to any subbundle D is also positive definite and hence regular. Thus, nonholonomic mechanical systems are always regular.

Proposition 3.13. *Conditions (3) and (4) in Theorem 3.12 are equivalent, respectively, to*

- (3') $\mathcal{T}^E E|_D = \mathcal{T}^E D \oplus F,$
- (4') $\mathcal{T}^E E|_D = \mathcal{T}^D D \oplus (\mathcal{T}^D D)^\perp.$

Proof. The equivalence between (4) and (4') is obvious, since we are assuming that the free Lagrangian is regular, i.e., ω_L is symplectic. The equivalence of (3) and (3') follows by computing the dimension of the corresponding spaces. The ranks of $\mathcal{T}^E E$, $\mathcal{T}^E D$ and F are

$$\begin{aligned} \text{rank}(\mathcal{T}^E E) &= 2 \text{rank}(E), \\ \text{rank}(\mathcal{T}^E D) &= \text{rank}(E) + \text{rank}(D), \\ \text{rank}(F) &= \text{rank}(D^\circ) = \text{rank}(E) - \text{rank}(D). \end{aligned}$$

Thus $\text{rank}(\mathcal{T}^E E) = \text{rank}(\mathcal{T}^E D) + \text{rank}(F)$, and the result follows. \square

3.3. Projectors. We can express the constrained dynamical section in terms of the free dynamical section by projecting to the adequate space, either $\mathcal{T}^E D$ or $\mathcal{T}^D D$, according to each of the above decompositions of $\mathcal{T}^E E|_D$. Of course, both procedures give the same result.

Projection to $\mathcal{T}^E D$. Assuming that the constrained system is regular, we have a direct sum decomposition

$$\mathcal{T}_a^E E = \mathcal{T}_a^E D \oplus F_a,$$

for every $a \in D$, where we recall that the subbundle $F \subset \mathcal{T}^E D$ is defined by $F = \omega_L^{-1}(\widetilde{D^\circ})$, or equivalently $\widetilde{D^\circ} = \omega_L(F)$.

Let us denote by P and Q the complementary projectors defined by this decomposition, that is,

$$P_a: \mathcal{T}_a^E E \rightarrow \mathcal{T}_a^E D \quad \text{and} \quad Q_a: \mathcal{T}_a^E E \rightarrow F_a, \quad \text{for all } a \in D.$$

Then we have,

Theorem 3.14. *Let (L, D) be a regular constrained Lagrangian system and let Γ_L be the solution of the free dynamics, i.e., $i_{\Gamma_L} \omega_L = dE_L$. Then the solution of the constrained dynamics is the SODE $\Gamma_{(L,D)}$ obtained by projection $\Gamma_{(L,D)} = P(\Gamma_L|_D)$.*

Proof. Indeed, if we write $\Gamma_{(L,D)}(a) = \Gamma_L(a) - Q(\Gamma_L(a))$ for $a \in D$, then we have

$$i_{\Gamma_{(L,D)}(a)} \omega_L - dE_L(a) = i_{\Gamma_L(a)} \omega_L - i_{Q(\Gamma_L(a))} \omega_L - dE_L(a) = -i_{Q(\Gamma_L(a))} \omega_L \in \widetilde{D_{\tau(a)}^\circ},$$

which is an element of $\widetilde{D_{\tau(a)}^\circ}$ because $Q(\Gamma_L(a))$ is in F_a . Moreover, since Γ_L is a SODE and $Q(\Gamma_L)$ is vertical (since it is in F), we have that $\Gamma_{(L,D)}$ is also a SODE. \square

We consider adapted local coordinates (x^i, y^a, y^A) corresponding to the choice of an adapted basis of sections $\{e_a, e_A\}$, where $\{e_a\}$ generate D . The annihilator D° of D is generated by $\{e^A\}$, and thus $\widetilde{D^\circ}$ is generated by $\{\mathcal{X}^A\}$. A simple calculation shows that a basis $\{Z_A\}$ of local sections of F is given by

$$Z_A = \mathcal{V}_A - Q_A^a \mathcal{V}_a, \quad (3.9)$$

where $Q_A^a = W_{Ab} \mathcal{C}^{ab}$ and \mathcal{C}^{ab} are the components of the inverse of the matrix \mathcal{C}_{ab} given by equation (3.6). The local expression of the projector over F is then

$$Q = Z_A \otimes \mathcal{V}^A.$$

If the expression of the free dynamical section Γ_L in this local coordinates is

$$\Gamma_L = y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha,$$

(where f^α are given by equation (2.7)), then the expression of the constrained dynamical section is

$$\Gamma_{(L,D)} = y^a \mathcal{X}_a + (f^a + f^A Q_A^a) \mathcal{V}_a,$$

where all the functions f^α are evaluated at $y^A = 0$.

Projection to $\mathcal{T}^D D$. We have seen that the regularity condition for the constrained system (L, D) can be equivalently expressed by requiring that the subbundle $\mathcal{T}^D D$ is a symplectic subbundle of $(\mathcal{T}^E E, \omega_L)$. It follows that, for every $a \in D$, we have a direct sum decomposition

$$\mathcal{T}_a^E E = \mathcal{T}_a^D D \oplus (\mathcal{T}_a^D D)^\perp.$$

Let us denote by \bar{P} and \bar{Q} the complementary projectors defined by this decomposition, that is,

$$\bar{P}_a: \mathcal{T}_a^E E \rightarrow \mathcal{T}_a^D D \quad \text{and} \quad \bar{Q}_a: \mathcal{T}_a^E E \rightarrow (\mathcal{T}_a^D D)^\perp, \quad \text{for all } a \in D.$$

Then, we have the following result:

Theorem 3.15. *Let (L, D) be a regular constrained Lagrangian system and let Γ_L be the solution of the free dynamics, i.e., $i_{\Gamma_L}\omega_L = dE_L$. Then the solution of the constrained dynamics is the SODE $\Gamma_{(L,D)}$ obtained by projection $\Gamma_{(L,D)} = \bar{P}(\Gamma_L|_D)$.*

Proof. From Theorem 3.14 we have that the solution $\Gamma_{(L,D)}$ of the constrained dynamics is related to the free dynamics by $\Gamma_L|_D = \Gamma + Q(\Gamma_L|_D)$. Let us prove that $Q(\Gamma_L|_D)$ takes values in $(\mathcal{T}^D D)^\perp$. Indeed, $Q(\Gamma_L|_D)$ takes values in $F = (\mathcal{T}^D D)^\perp \cap \text{Ver}(\mathcal{T}^E E)$, so that, in particular, it takes values in $(\mathcal{T}^D D)^\perp$. Thus, since Γ is a section of $\mathcal{T}^D D$, it follows that $\Gamma_L|_D = \Gamma_{(L,D)} + Q(\Gamma_L|_D)$ is a decomposition of $\Gamma_L|_D$ according to $\mathcal{T}^E E|_D = \mathcal{T}^D D \oplus (\mathcal{T}^D D)^\perp$, which implies $\Gamma_{(L,D)} = \bar{P}(\Gamma_L|_D)$. \square

In adapted coordinates, a local basis of sections of $(\mathcal{T}^D D)^\perp$ is $\{Y_A, Z_A\}$, where the sections Z_A are given by (3.9) and the sections Y_A are

$$Y_A = \mathcal{X}_A - Q_A^a \mathcal{X}_a + \mathcal{C}^{bc}(M_{Ab} - M_{ab}Q_A^a)\mathcal{V}_c,$$

with $M_{\alpha\beta} = \omega_L(\mathcal{X}_\alpha, \mathcal{X}_\beta)$. Therefore the expression of the projector onto $(\mathcal{T}^D D)^\perp$ is

$$\bar{Q} = Z_A \otimes \mathcal{V}^A + Y_A \otimes \mathcal{X}^A.$$

Note that $S(Y_A) = Z_A$.

3.4. The distributional approach. The equations for the Lagrange-d'Alembert section Γ can be entirely written in terms of objects in the manifold $\mathcal{T}^D D$. Recall that $\mathcal{T}^D D$ is not a Lie algebroid. In order to do this, define the 2-section $\omega^{L,D}$ as the restriction of ω_L to $\mathcal{T}^D D$. If (L, D) is regular, then $\mathcal{T}^D D$ is a symplectic subbundle of $(\mathcal{T}^E E, \omega_L)$. From this, it follows that $\omega^{L,D}$ is a symplectic section on that bundle. We also define $\varepsilon^{L,D}$ to be the restriction of dE_L to $\mathcal{T}^D D$. Then, taking the restriction of Lagrange-d'Alembert equations to $\mathcal{T}^D D$, we get the following equation

$$i_\Gamma \omega^{L,D} = \varepsilon^{L,D}, \quad (3.10)$$

which uniquely determines the section Γ . Indeed, the unique solution Γ of the above equations is the solution of Lagrange-d'Alembert equations: if we denote by λ the constraint force, we have for every $u \in \mathcal{T}_a^D D$ that

$$\omega_L(\Gamma(a), u) - \langle dE_L(a), u \rangle = \langle \lambda(a), \mathcal{T}\tau(u) \rangle = 0,$$

where we have taken into account that $\mathcal{T}\tau(u) \in D$ and $\lambda(a) \in D^\circ$.

This approach, the so called **distributional approach**, was initiated by Bocharov and Vinogradov (see [67]) and further developed by Śniatycki and coworkers [2, 24, 62]. Similar equations, within the framework of Lie algebroids, are the base of the theory proposed in [56].

Remark 3.16 (Alternative description with $\mathcal{T}^E D$). One can also consider the restriction to $\mathcal{T}^E D$, which is a Lie algebroid, but no further simplification is achieved by this. If $\bar{\omega}$ is the restriction of ω_L to $\mathcal{T}^E D$ and $\bar{\varepsilon}$ is the restriction of dE_L to $\mathcal{T}^E D$, then the Lagrange-d'Alembert equations can be written in the form $i_\Gamma \bar{\omega} - \bar{\varepsilon} = \bar{\lambda}$, where $\bar{\lambda}$ is the restriction of the constraint force to $\mathcal{T}^E D$, which, in general, does not vanish. Also notice that the 2-form $\bar{\omega}$ is closed but, in general, degenerated. \bullet

3.5. The nonholonomic bracket. Let f, g be two smooth functions on D and take arbitrary extensions to E denoted by the same letters (if there is no possibility of confusion). Suppose that X_f and X_g are the Hamiltonian sections on $\mathcal{T}^E E$ given respectively by

$$i_{X_f} \omega_L = df \quad \text{and} \quad i_{X_g} \omega_L = dg.$$

We define the **nonholonomic bracket** of f and g as follows:

$$\{f, g\}_{nh} = \omega_L(\bar{P}(X_f), \bar{P}(X_g)). \quad (3.11)$$

Note that if f' is another extension of f , then $(X_f - X_{f'})|_D$ is a section of $(\mathcal{T}^D D)^\perp$ and, thus, we deduce that (3.11) does not depend on the chosen extensions. The nonholonomic bracket is an almost-Poisson bracket, i.e., it is skew-symmetric, a derivation in each argument with respect to the usual product of functions and does not satisfy the Jacobi identity.

In addition, one can prove the following formula

$$\dot{f} = \{f, E_L\}_{nh}. \quad (3.12)$$

Indeed, we have

$$\begin{aligned} \dot{f} &= d_{\Gamma(L,D)} f = i_{\Gamma(L,D)} df = i_{\Gamma(L,D)} i_{X_f} \omega_L \\ &= \omega_L(X_f, \Gamma(L,D)) = \omega_L(X_f, \bar{P}(\Gamma_L)) \\ &= \omega_L(\bar{P}(X_f), \bar{P}(\Gamma_L)) = \{f, E_L\}_{nh}. \end{aligned}$$

Equation (3.12) implies once more the conservation of the energy (by the skew-symmetric character of the nonholonomic bracket).

Alternatively, since $\mathcal{T}^D D$ is an anchored vector bundle, one can take the function $f \in C^\infty(D)$ and its differential $\bar{d}f \in \text{Sec}((\mathcal{T}^D D)^*)$. Since $\omega^{L,D}$ is regular, we have a unique section $\bar{X}_f \in \text{Sec}(\mathcal{T}^D D)$ defined by $i_{\bar{X}_f} \omega^{L,D} = \bar{d}f$. Then the nonholonomic bracket of two functions f and g is $\{f, g\}_{nh} = \omega^{L,D}(\bar{X}_f, \bar{X}_g)$. Note that if $\tilde{f} \in C^\infty(E)$ (resp. $\tilde{g} \in C^\infty(E)$) is an extension to E of f (resp., g), then $\bar{X}_f = \bar{P}(X_{\tilde{f}})|_D$ (resp., $\bar{X}_g = \bar{P}(X_{\tilde{g}})|_D$).

4. MORPHISMS AND REDUCTION

One important advantage of dealing with Lagrangian systems evolving on Lie algebroids is that the reduction procedure can be naturally handled by considering morphisms of Lie algebroids, as it was already observed by Weinstein [68]. We study in this section the transformation laws of the different geometric objects in our theory and we apply these results to the study of the reduction theory.

Proposition 4.1. *Let $\Phi: E \rightarrow E'$ be a morphism of Lie algebroids, and consider the Φ -tangent prolongation of Φ , i.e. $\mathcal{T}^\Phi \Phi: \mathcal{T}^E E \rightarrow \mathcal{T}^{E'} E'$. Let ξ^V and ξ'^V , S and S' , and Δ and Δ' , be the vertical liftings, the vertical endomorphisms, and the Liouville sections on E and E' , respectively. Then,*

- (1) $\mathcal{T}^\Phi \Phi(\xi^V(a, b)) = \xi'^V(\Phi(a), \Phi(b))$, for all $(a, b) \in E \times_M E$,
- (2) $\mathcal{T}^\Phi \Phi \circ \Delta = \Delta' \circ \Phi$,
- (3) $\mathcal{T}^\Phi \Phi \circ S = S' \circ \mathcal{T}^\Phi \Phi$.

Proof. For the first property, we notice that both terms are vertical, so that we just have to show that their action on functions coincide. For every function $f' \in C^\infty(E')$, we deduce that

$$\begin{aligned} \rho'^1(\mathcal{T}^\Phi \Phi(\xi^V(a, b)))f' &= T\Phi(\rho^1(\xi^V(a, b)))f' = T\Phi(b_a^V)f' = b_a^V(f' \circ \Phi) \\ &= \frac{d}{dt} f'(\Phi(a + tb)) \Big|_{t=0} = \frac{d}{dt} f'(\Phi(a) + t\Phi(b)) \Big|_{t=0} \\ &= \Phi(b)_{\Phi(a)}^V(f') = \rho'^1(\xi'^V(\Phi(a), \Phi(b)))f'. \end{aligned}$$

For the second property, we have $\Delta(a) = \xi^V(a, a)$ so that applying the first property it follows that

$$\mathcal{T}^\Phi \Phi(\Delta(a)) = \mathcal{T}^\Phi \Phi(\xi^V(a, a)) = \xi'^V(\Phi(a), \Phi(a)) = \Delta'(\Phi(a)).$$

Finally, for any $z = (a, b, V) \in \mathcal{T}^E E$, we obtain that

$$\begin{aligned} \mathcal{T}^\Phi \Phi(S(z)) &= \mathcal{T}^\Phi \Phi(\xi^V(a, b)) = \xi'^V(\Phi(a), \Phi(b)) \\ &= S'(\Phi(a), \Phi(b), T\Phi(V)) = S'(\mathcal{T}^\Phi \Phi(z)), \end{aligned}$$

which concludes the proof. \square

Proposition 4.2. *Let $L \in C^\infty(E)$ be a Lagrangian function, θ_L the Cartan form and $\omega_L = -d\theta_L$. Let $\Phi: E \rightarrow E'$ be a Lie algebroid morphism and suppose that $L = L' \circ \Phi$, with $L' \in C^\infty(E')$ a Lagrangian function. Then, we have*

- (1) $(\mathcal{T}^\Phi \Phi)^* \theta_{L'} = \theta_L$,
- (2) $(\mathcal{T}^\Phi \Phi)^* \omega_{L'} = \omega_L$,
- (3) $(\mathcal{T}^\Phi \Phi)^* E_{L'} = E_L$,
- (4) $G_{\Phi(a)}^{L'}(\Phi(b), \Phi(c)) = G_a^L(b, c)$, for every $a \in E$ and every $b, c \in E_{\tau(a)}$.

Proof. Indeed, for every $Z \in \mathcal{T}^E E$ we have

$$\begin{aligned} \langle (\mathcal{T}^\Phi \Phi)^* \theta_{L'}, Z \rangle &= \langle \theta_{L'}, \mathcal{T}^\Phi \Phi(Z) \rangle = \langle dL', S'(\mathcal{T}^\Phi \Phi(Z)) \rangle = \langle dL', \mathcal{T}^\Phi \Phi(S(Z)) \rangle \\ &= \langle (\mathcal{T}^\Phi \Phi)^* dL', S(Z) \rangle = \langle d(\mathcal{T}^\Phi \Phi)^* L', S(Z) \rangle = \langle d(L' \circ \Phi), S(Z) \rangle \\ &= \langle dL, S(Z) \rangle = \langle \theta_L, Z \rangle, \end{aligned}$$

where we have used the transformation rule for the vertical endomorphism. The second property follows from the fact that $\mathcal{T}^\Phi \Phi$ is a morphism, so that $(\mathcal{T}^\Phi \Phi)^* d = d(\mathcal{T}^\Phi \Phi)^*$. The third one follows similarly and the fourth is a consequence of the second property and the definitions of the tensors G^L and $G^{L'}$. \square

Let Γ be a SODE and $L \in C^\infty(E)$ be a Lagrangian. For convenience, we define the 1-form $\delta_\Gamma L \in \text{Sec}((\mathcal{T}^E E)^*)$ by

$$\langle \delta_\Gamma L, Z \rangle = \langle dE_L - i_\Gamma \omega_L, Z \rangle = \langle dE_L, Z \rangle - \omega_L(\Gamma, Z),$$

for every section Z of $\mathcal{T}^E E$. We notice that Γ is the solution of the free dynamics if and only if $\delta_\Gamma L = 0$. On the other hand, notice that the 1-form $\delta_\Gamma L$ is semibasic, because Γ is a SODE.

Proposition 4.3. *Let Γ be a SODE in E and Γ' a SODE in E' . Let $L \in C^\infty(E)$ and $L' \in C^\infty(E')$ be Lagrangian functions defined on E and E' , respectively, such that $L = L' \circ \Phi$. Then,*

$$\langle \delta_\Gamma L - (\mathcal{T}^\Phi \Phi)^* \delta_{\Gamma'} L', Z \rangle = \omega_{L'}(\Gamma' \circ \Phi - \mathcal{T}^\Phi \Phi \circ \Gamma, \mathcal{T}^\Phi \Phi(Z)), \quad (4.1)$$

for every section Z of $\mathcal{T}^E E$.

Proof. Indeed, from $(\mathcal{T}^\Phi \Phi)^* dE_{L'} = dE_L$, we have that

$$\begin{aligned} \langle \delta_\Gamma L - (\mathcal{T}^\Phi \Phi)^* \delta_{\Gamma'} L', Z \rangle &= \langle (\mathcal{T}^\Phi \Phi)^* i_{\Gamma'} \omega_{L'} - i_\Gamma \omega_L, Z \rangle \\ &= \langle (\mathcal{T}^\Phi \Phi)^* i_{\Gamma'} \omega_{L'} - i_\Gamma (\mathcal{T}^\Phi \Phi)^* \omega_{L'}, Z \rangle \\ &= \omega_{L'}(\Gamma' \circ \Phi - \mathcal{T}^\Phi \Phi \circ \Gamma, \mathcal{T}^\Phi \Phi(Z)), \end{aligned}$$

which concludes the proof. \square

4.1. Reduction of the free dynamics. Here, we build on Propositions 4.2 and 4.3 to identify conditions under which the dynamics can be reduced under a morphism of Lie algebroids. We first notice that, from Proposition 4.2, if Φ is fiberwise surjective morphism and L is a regular Lagrangian on E , then L' is a regular Lagrangian on E' (note that $\mathcal{T}^\Phi \Phi: \mathcal{T}^E E \rightarrow \mathcal{T}^{E'} E'$ is a fiberwise surjective morphism). Thus, the dynamics of both systems is uniquely defined.

Theorem 4.4 (Reduction of the free dynamics). *Suppose that the Lagrangian functions L and L' are Φ -related, that is, $L = L' \circ \Phi$. If Φ is a fiberwise surjective morphism and L is a regular Lagrangian then L' is also a regular Lagrangian. Moreover, if Γ_L and $\Gamma_{L'}$ are the solutions of the free dynamics defined by L and L' then*

$$\mathcal{T}^\Phi \Phi \circ \Gamma_L = \Gamma_{L'} \circ \Phi.$$

Therefore, if $a(t)$ is a solution of the free dynamics defined by L , then $\Phi(a(t))$ is a solution of the free dynamics defined by L' .

Proof. If Γ_L and $\Gamma_{L'}$ are the solutions of the dynamics, then $\delta_{\Gamma_L} L = 0$ and $\delta_{\Gamma_{L'}} L' = 0$ so that the left-hand side in equation (4.1) vanishes. Thus

$$\omega_{L'}(\Gamma_{L'} \circ \Phi - \mathcal{T}^\Phi \Phi \circ \Gamma_L, \mathcal{T}^\Phi \Phi(Z)) = 0,$$

for every $Z \in \text{Sec}(\mathcal{T}^E E)$. Therefore, using that L' is regular and the fact that $\mathcal{T}^\Phi \Phi$ is a fiberwise surjective morphism, we conclude the result. \square

We will say that the unconstrained dynamics $\Gamma_{L'}$ is the **reduction of the unconstrained dynamics** Γ_L by the morphism Φ .

4.2. Reduction of the constrained dynamics. The above results about reduction of unconstrained Lagrangian systems can be easily generalized to nonholonomic constrained Lagrangian systems whenever the constraints of one system are mapped by the morphism to the constraints of the second system. Let us elaborate on this.

Let (L, D) be a constrained Lagrangian system on a Lie algebroid E and let (L', D') be another constrained Lagrangian system on a second Lie algebroid E' . Along this section, we assume that there is a fiberwise surjective morphism of Lie algebroids $\Phi: E \rightarrow E'$ such that $L = L' \circ \Phi$ and $\Phi(D) = D'$. The latter condition implies that the base map is also surjective, so that we will assume that Φ is an epimorphism (i.e., in addition to being fiberwise surjective, the base map φ is a submersion).

As a first consequence, we have $G_{\Phi(a)}^{L', D'}(\Phi(b), \Phi(c)) = G_a^{L, D}(b, c)$, for every $a \in D$ and every $b, c \in D_{\pi(a)}$, and therefore, if (L, D) is regular, then so is (L', D') .

Lemma 4.5. *With respect to the decompositions $\mathcal{T}^E E|_D = \mathcal{T}^E D \oplus F$ and $\mathcal{T}^{E'} E'|_{D'} = \mathcal{T}^{E'} D' \oplus F'$, we have the following properties:*

- (1) $\mathcal{T}^\Phi \Phi(\mathcal{T}^E D) = \mathcal{T}^{E'} D'$,
- (2) $\mathcal{T}^\Phi \Phi(F) = F'$,
- (3) *If P, Q and P', Q' are the projectors associated with (L, D) and (L', D') , respectively, then $P' \circ \mathcal{T}^\Phi \Phi = \mathcal{T}^\Phi \Phi \circ P$ and $Q' \circ \mathcal{T}^\Phi \Phi = \mathcal{T}^\Phi \Phi \circ Q$.*

With respect to the decompositions $\mathcal{T}^E E|_D = \mathcal{T}^D D \oplus (\mathcal{T}^D D)^\perp$ and $\mathcal{T}^{E'} E'|_{D'} = \mathcal{T}^{D'} D' \oplus (\mathcal{T}^{D'} D')^\perp$ we have the following properties:

- (4) $\mathcal{T}^\Phi \Phi(\mathcal{T}^D D) = \mathcal{T}^{D'} D'$,
- (5) $\mathcal{T}^\Phi \Phi((\mathcal{T}^D D)^\perp) = (\mathcal{T}^{D'} D')^\perp$,
- (6) *If \bar{P}, \bar{Q} and \bar{P}', \bar{Q}' are the projectors associated with (L, D) and (L', D') , respectively, then $\bar{P}' \circ \mathcal{T}^\Phi \Phi = \mathcal{T}^\Phi \Phi \circ \bar{P}$ and $\bar{Q}' \circ \mathcal{T}^\Phi \Phi = \mathcal{T}^\Phi \Phi \circ \bar{Q}$.*

Proof. From the definition of $\mathcal{T}^\Phi \Phi$, it follows that

$$(\mathcal{T}^\Phi \Phi)(\mathcal{T}^E D) \subseteq \mathcal{T}^{E'} D', \quad (\mathcal{T}^\Phi \Phi)(\mathcal{T}^D D) \subseteq \mathcal{T}^{D'} D'.$$

Thus, one may consider the vector bundle morphisms

$$\mathcal{T}^\Phi \Phi: \mathcal{T}^E D \rightarrow \mathcal{T}^{E'} D', \quad \mathcal{T}^\Phi \Phi: \mathcal{T}^D D \rightarrow \mathcal{T}^{D'} D'.$$

Moreover, using that Φ is fiberwise surjective and that φ is a submersion, we deduce that the rank of the above morphisms is maximum. This proves (1) and (4).

The proof of (5) is as follows. For every $a' \in D'$, one can choose $a \in D$ such that $\Phi(a) = a'$, and one can write any element $w' \in \mathcal{T}_{a'}^{D'} D'$ as $w' = \mathcal{T}^\Phi \Phi(w)$ for some $w \in \mathcal{T}_a^D D$. Thus, if $z \in (\mathcal{T}_a^D D)^\perp$, for every $w' \in \mathcal{T}_{a'}^{D'} D'$ we have

$$\omega_{L'}(\mathcal{T}^\Phi \Phi(z), w') = \omega_{L'}(\mathcal{T}^\Phi \Phi(z), \mathcal{T}^\Phi \Phi(w)) = \omega_L(z, w) = 0,$$

from where it follows that $\mathcal{T}^\Phi \Phi(z) \in (\mathcal{T}^{D'} D')^\perp$. In a similar way, using that $\mathcal{T}^\Phi \Phi : (\mathcal{T}^E E)|_D \rightarrow (\mathcal{T}^{E'} E')|_{D'}$ is fiberwise surjective, (2) in Proposition 4.2 and (4), we obtain that $(\mathcal{T}^{D'} D')^\perp \subseteq (\mathcal{T}^\Phi \Phi)((\mathcal{T}^D D)^\perp)$.

For the proof of (2) we have that

$$\mathcal{T}^\Phi \Phi(F) = \mathcal{T}^\Phi \Phi((\mathcal{T}^D D)^\perp \cap \text{Ver}(\mathcal{T}^E E)) \subseteq (\mathcal{T}^{D'} D')^\perp \cap \text{Ver}(\mathcal{T}^{E'} E') = F'.$$

Thus, using that $\mathcal{T}^\Phi \Phi : (\mathcal{T}^E E)|_D \rightarrow (\mathcal{T}^{E'} E')|_{D'}$ is fiberwise surjective, the fact that $(\mathcal{T}^E E)|_D = \mathcal{T}^E D \oplus F$ and (1), it follows that

$$(\mathcal{T}^{E'} E')|_{D'} = \mathcal{T}^{E'} D' \oplus (\mathcal{T}^\Phi \Phi)(F).$$

Therefore, since $(\mathcal{T}^{E'} E')|_{D'} = \mathcal{T}^{E'} D' \oplus F'$, we conclude that (2) holds.

Finally, (3) is an immediate consequence of (1) and (2), and similarly, (6) is an immediate consequence of (4) and (5). \square

From the properties above, we get the following result.

Theorem 4.6 (Reduction of the constrained dynamics). *Let (L, D) be a regular constrained Lagrangian system on a Lie algebroid E and let (L', D') be a constrained Lagrangian system on a second Lie algebroid E' . Assume that a fiberwise surjective morphism of Lie algebroids $\Phi : E \rightarrow E'$ exists such that $L = L' \circ \Phi$ and $\Phi(D) = D'$. If $\Gamma_{(L, D)}$ is the constrained dynamics for L and $\Gamma_{(L', D')}$ is the constrained dynamics for L' , then $\mathcal{T}^\Phi \Phi \circ \Gamma_{(L, D)} = \Gamma_{(L', D')} \circ \Phi$. If $a(t)$ is a solution of Lagrange-d'Alembert differential equations for L , then $\Phi(a(t))$ is a solution of Lagrange-d'Alembert differential equations for L' .*

Proof. For the free dynamics, we have that $\mathcal{T}^\Phi \Phi \circ \Gamma_L = \Gamma_{L'} \circ \Phi$. Moreover, from property (3) in Lemma 4.5, for every $a \in D$, we have that

$$\begin{aligned} \mathcal{T}^\Phi \Phi(\Gamma_{(L, D)}(a)) &= \mathcal{T}^\Phi \Phi(P(\Gamma_L(a))) = P'(\mathcal{T}^\Phi \Phi(\Gamma_L(a))) \\ &= P'(\Gamma_{L'}(\Phi(a))) = \Gamma_{(L', D')}(\Phi(a)), \end{aligned}$$

which concludes the proof. \square

We will say that the constrained dynamics $\Gamma_{(L', D')}$ is the **reduction of the constrained dynamics** $\Gamma_{(L, D)}$ by the morphism Φ .

Theorem 4.7. *Under the same hypotheses as in Theorem 4.6, we have that*

$$\{f' \circ \Phi, g' \circ \Phi\}_{nh} = \{f', g'\}'_{nh} \circ \Phi,$$

for $f', g' \in C^\infty(D')$, where $\{\cdot, \cdot\}_{nh}$ (respectively, $\{\cdot, \cdot\}'_{nh}$) is the nonholonomic bracket for the constrained system (L, D) (respectively, (L', D')). In other words, $\Phi : D \rightarrow D'$ is an almost-Poisson morphism.

Proof. Using (2) in Proposition 4.2 and the fact that Φ is a Lie algebroid morphism, we deduce that

$$(i_{X_{f' \circ \Phi}}(\mathcal{T}^\Phi \Phi)^* \omega_{L'}) = i_{X_{f'}} \omega_{L'} \circ \Phi.$$

Thus, since $\mathcal{T}^\Phi \Phi$ is fiberwise surjective, we obtain that

$$\mathcal{T}^\Phi \Phi \circ X_{f' \circ \Phi} = X_{f'} \circ \Phi.$$

Now, from (3.11) and Lemma 4.5, we conclude that

$$\{f' \circ \Phi, g' \circ \Phi\}_{nh} = \{f', g'\}'_{nh} \circ \Phi.$$

\square

One of the most important cases in the theory of reduction is the case of reduction by a symmetry group. In this respect, we have the following result.

Theorem 4.8 ([43, 48]). *Let $q_G^Q: Q \rightarrow M$ be a principal G -bundle, let $\tau: E \rightarrow Q$ be a Lie algebroid, and assume that we have an action of G on E such that the quotient vector bundle E/G is well-defined. If the set $\text{Sec}(E)^G$ of equivariant sections of E is a Lie subalgebra of $\text{Sec}(E)$, then the quotient $E' = E/G$ has a canonical Lie algebroid structure over M such that the canonical projection $q_G^E: E \rightarrow E/G$, given by $a \mapsto [a]_G$, is a (fiberwise bijective) Lie algebroid morphism over q_G^Q .*

As a concrete example of application of the above theorem, we have the well-known case of the Atiyah or Gauge algebroid. In this case, the Lie algebroid E is the standard Lie algebroid $TQ \rightarrow Q$, the action is by tangent maps $gv \equiv T\psi_g(v)$, the reduction is the Atiyah Lie algebroid $TQ/G \rightarrow Q/G$ and the quotient map $q_G^{TQ}: TQ \rightarrow TQ/G$ is a Lie algebroid epimorphism. It follows that if L is a G -invariant regular Lagrangian on TQ then the unconstrained dynamics for L projects to the unconstrained dynamics for the reduced Lagrangian L' . Moreover, if the constraints D are also G -invariant, then the constrained dynamics for (L, D) reduces to the constrained dynamics for $(L', D/G)$.

On a final note, we mention that the pullback of the distributional equation $i_{\Gamma'}\omega^{L',D'} - \varepsilon^{L',D'} = 0$ by $\mathcal{T}^\Phi\Phi$ is precisely $(i_\Gamma\omega^{L,D} - \varepsilon^{L,D}) \circ \mathcal{T}^\Phi\Phi = 0$.

4.3. Reduction by stages. As a direct consequence of the results exposed above, one can obtain a theory of reduction by stages. In Poisson geometry, reduction by stages is a straightforward procedure. Given the fact that the Lagrangian counterpart of Poisson reduction is Lagrangian reduction, it is not strange that reduction by stages in our framework becomes also straightforward.

The Lagrangian theory of reduction by stages is a consequence of the following basic observation:

Let $\Phi_1: E_0 \rightarrow E_1$ and $\Phi_2: E_1 \rightarrow E_2$ be a fiberwise surjective morphisms of Lie algebroids and let $\Phi: E_0 \rightarrow E_2$ be the composition $\Phi = \Phi_2 \circ \Phi_1$. The reduction of a Lagrangian system in E_0 by Φ can be obtained by first reducing by Φ_1 and then reducing the resulting Lagrangian system by Φ_2 .

This result follows using that $\mathcal{T}^\Phi\Phi = \mathcal{T}^{\Phi_2}\Phi_2 \circ \mathcal{T}^{\Phi_1}\Phi_1$. Based on this fact, one can analyze one of the most interesting cases of reduction: the reduction by the action of a symmetry group. We consider a group G acting on a manifold Q and a closed normal subgroup N of G . The process of reduction by stages is illustrated in the following diagram

$$\begin{array}{ccc}
 & & \tau_Q: E_0 = TQ \rightarrow M_0 = Q \\
 & & \downarrow \text{First reduction } \cdot/N \\
 \text{Total reduction } \cdot/G & \left(\begin{array}{c} \curvearrowright \end{array} \right. & \tau_1: E_1 = TQ/N \rightarrow M_1 = Q/N \\
 & & \downarrow \text{Second reduction } \cdot/(G/N) \\
 & & \tau_2: E_2 = (TQ/N)/(G/N) \rightarrow M_2 = (Q/N)/(G/N)
 \end{array}$$

In order to prove our results about reduction by stages, we have to prove that E_0, E_1 and E_2 are Lie algebroids, that the quotient maps $\Phi_1: E_0 \rightarrow E_1$, $\Phi_2: E_1 \rightarrow E_2$ and $\Phi: E_0 \rightarrow E_2$ are Lie algebroids morphisms and that the composition $\Phi_1 \circ \Phi_2$ equals to Φ . Our proof is based on the following well-known result (see [15]), which contains most of the ingredients in the theory of reduction by stages.

Theorem 4.9. ([15]) *Let $q_G^Q: Q \rightarrow M$ be a principal G -bundle and N a closed normal subgroup of G . Then,*

- (1) $q_N^Q: Q \rightarrow Q/N$ is a principal N -bundle,

- (2) G/N acts on Q/N by the rule $[g]_N[q]_N = [gq]_N$,
- (3) $q_{G/N}^Q: Q/N \rightarrow (Q/N)/(G/N)$ is a principal (G/N) -bundle.
- (4) The map $i: Q/G \rightarrow (Q/N)/(G/N)$ defined by $[q]_G \mapsto [[q]_N]_{G/N}$ is a diffeomorphism.

Building on the previous results, one can deduce the following theorem, which states that the reduction of a Lie algebroid can be done by stages.

Theorem 4.10. *Let $q_G^Q: Q \rightarrow M$ be a principal G -bundle and N be a closed normal subgroup of G . Then,*

- (1) $\tau_{TQ/G}: TQ/G \rightarrow Q/G$ is a Lie algebroid and $q_G^{TQ}: TQ \rightarrow TQ/G$ is a Lie algebroid epimorphism,
- (2) $\tau_{TQ/N}: TQ/N \rightarrow Q/N$ is a Lie algebroid and $q_N^{TQ}: TQ \rightarrow TQ/N$ is a Lie algebroid epimorphism,
- (3) G/N acts on TQ/N by the rule $[g]_N[v]_N = [gv]_N$,
- (4) $\tau_{(TQ/N)/(G/N)}: (TQ/N)/(G/N) \rightarrow (Q/N)/(G/N)$ is a Lie algebroid and $q_{G/N}^{TQ/N}: TQ/N \rightarrow (TQ/N)/(G/N)$ is a Lie algebroid epimorphism,
- (5) The map $I: TQ/G \rightarrow (TQ/N)/(G/N)$ defined by $[v]_G \mapsto [[v]_N]_{G/N}$ is an isomorphism of Lie algebroids over the map i .

Proof. The vector bundle $\tau_{TQ/G}: TQ/G \rightarrow Q/G$ (respectively, $\tau_{TQ/N}: TQ/N \rightarrow Q/N$) is the Atiyah algebroid for the principal G -bundle $q_G^Q: Q \rightarrow Q/G$ (respectively, $q_N^Q: Q \rightarrow Q/N$), so that (1) and (2) are obvious. Condition (3) is just condition (2) of Theorem 4.9 applied to the principal N -bundle $TQ \rightarrow TQ/N$. To prove condition (4), we notice that the action of G/N on the Lie algebroid TQ/N is free and satisfies the conditions of Theorem 4.8. Finally, the Lie algebroid morphism $j: TQ \rightarrow TQ/N$ is equivariant with respect to the G -action on TQ and the (G/N) -action on TQ/N . Thus, it induces a morphism of Lie algebroids in the quotient. It is an isomorphism since it is a diffeomorphism by Theorem 4.9. \square

The following diagram illustrates the above situation:

$$\begin{array}{ccccc}
 & & \Phi & & \\
 & \curvearrowright & & \curvearrowleft & \\
 TQ & \xrightarrow{\Phi_1} & TQ/N & \xrightarrow{\Phi_2} & (TQ/N)/(G/N) \\
 \tau_Q \downarrow & & \downarrow \tau_1 & & \downarrow \tau_2 \\
 Q & \xrightarrow{q_N^Q} & Q/N & \xrightarrow{q_{G/N}^Q} & (Q/N)/(G/N) \\
 & \curvearrowleft & & \curvearrowright & \\
 & & q_G^Q & &
 \end{array}$$

In particular, for the unconstrained case one has the following result.

Theorem 4.11 (Reduction by stages of the free dynamics). *Let $q_G^Q: Q \rightarrow Q/G$ be a principal G -bundle, and N a closed normal subgroup of G . Let L be a Lagrangian function on Q which is G -invariant. Then the reduction by the symmetry group G can be performed in two stages:*

1. reduce by the normal subgroup N ,
2. reduce the resulting dynamics from 1. by the residual symmetry group G/N .

Since the dynamics of a constrained system is obtained by projection of the free dynamics, we also have the following result.

Theorem 4.12 (Reduction by stages of the constrained dynamics). *Let $q_G^Q: Q \rightarrow Q/G$ be a principal G -bundle and N a closed normal subgroup of G . Let (L, D) be*

a G -invariant constrained Lagrangian system. Then the reduction by the symmetry group G can be performed in two stages:

1. reduce by the normal subgroup N ,
2. reduce the resulting dynamics from 1. by the residual symmetry group G/N .

5. THE MOMENTUM EQUATION

In this section, we introduce the momentum map for a constrained system on a Lie algebroid, and examine its evolution along the dynamics. This gives rise to the so-called momentum equation.

5.1. Unconstrained case. Let us start by discussing the unconstrained case. Let $\tau_E : E \rightarrow M$ be a Lie algebroid over a manifold M and $L : E \rightarrow \mathbb{R}$ be a regular Lagrangian function. Suppose that $\tau_K : K \rightarrow M$ is a vector bundle over M and that $\Psi : K \rightarrow E$ is a vector bundle morphism (over the identity of M) between K and E . Then, we can define **the unconstrained momentum map** $J_{(L,\Psi)} : E \rightarrow K^*$ **associated with L and Ψ** as follows

$$J_{(L,\Psi)}(a) \in K_x^*, \text{ for } a \in E_x,$$

and

$$(J_{(L,\Psi)}(a))(k) = \frac{d}{dt}\bigg|_{t=0} L(a + t\Psi(k)) = \Psi(k)_a^\vee(L), \text{ for } k \in K_x.$$

If $\sigma : M \rightarrow K$ is a section of $\tau_K : K \rightarrow M$ then, using the momentum map $J_{(L,\Psi)}$, we may introduce the real function $J_{(L,\Psi)}^\sigma : E \rightarrow \mathbb{R}$ given by

$$J_{(L,\Psi)}^\sigma(a) = J_{(L,\Psi)}(a)(\sigma(x)) = \Psi(\sigma(x))_a^\vee(L), \text{ for } a \in E_x. \quad (5.1)$$

Theorem 5.1 (The unconstrained momentum equation). *Let Γ_L be the Euler-Lagrange section associated with the regular Lagrangian function $L : E \rightarrow \mathbb{R}$. If $\sigma : M \rightarrow K$ is a section of $\tau_K : K \rightarrow M$ and $(\Psi \circ \sigma)^c \in \text{Sec}(\mathcal{T}^E E)$ is the complete lift of $(\Psi \circ \sigma) \in \text{Sec}(E)$, we have that*

$$\langle d^{\mathcal{T}^E E} J_{(L,\Psi)}^\sigma, \Gamma_L \rangle = \langle d^{\mathcal{T}^E E} L, (\Psi \circ \sigma)^c \rangle, \quad (5.2)$$

where $d^{\mathcal{T}^E E}$ is the differential of Lie algebroid $\mathcal{T}^E E \rightarrow E$. In particular, if $\langle d^{\mathcal{T}^E E} L, (\Psi \circ \sigma)^c \rangle = 0$, then the real function $J_{(L,\Psi)}^\sigma$ is a constant of the motion for the Lagrangian dynamics associated with the Lagrangian function L .

Proof. Let $S : \mathcal{T}^E E \rightarrow \mathcal{T}^E E$ be the vertical endomorphism. If $(\Psi \circ \sigma)^v \in \text{Sec}(\mathcal{T}^E E)$ is the vertical lift of $(\Psi \circ \sigma) \in \text{Sec}(E)$ then, using (5.1) and the fact that $S(\Psi \circ \sigma)^c = (\Psi \circ \sigma)^v$, it follows that

$$J_{(L,\Psi)}^\sigma = \theta_L((\Psi \circ \sigma)^c), \quad (5.3)$$

where θ_L is the Cartan 1-form associated with L .

Thus, from (5.3), we deduce that

$$d^{\mathcal{T}^E E} J_{(L,\Psi)}^\sigma = \mathcal{L}_{(\Psi \circ \sigma)^c}^{\mathcal{T}^E E} \theta_L + i(\Psi \circ \sigma)^c(\omega_L),$$

ω_L being the Cartan 2-form associated with L .

Therefore, if $E_L : E \rightarrow \mathbb{R}$ is the Lagrangian energy, we obtain that

$$\begin{aligned} \langle d^{\mathcal{T}^E E} J_{(L,\Psi)}^\sigma, \Gamma_L \rangle &= \langle d^{\mathcal{T}^E E}(\theta_L(\Gamma_L)), (\Psi \circ \sigma)^c \rangle - \langle d^{\mathcal{T}^E E} E_L, (\Psi \circ \sigma)^c \rangle \\ &= \langle \theta_L, [(\Psi \circ \sigma)^c, \Gamma_L] \rangle. \end{aligned} \quad (5.4)$$

Now, from (2.11) and since Γ_L is a SODE section, it follows that

$$\Theta_L(\Gamma_L) = \langle d^{\mathcal{T}^E E} L, \Delta \rangle, \quad \langle \theta_L, [(\Psi \circ \sigma)^c, \Gamma_L] \rangle = 0,$$

where $\Delta \in \text{Sec}(\mathcal{T}^E E)$ is the Liouville section. Consequently, using (5.4) we deduce that (5.2) holds. \square

Remark 5.2 (Conservation of momentum on TM). Let $L : TM \rightarrow \mathbb{R}$ be a standard regular Lagrangian function on TM . Suppose that G is a Lie group with Lie algebra \mathfrak{g} and that $\psi : G \times M \rightarrow M$ is a (left) action of G on M . Then, we may consider the trivial vector bundle over M

$$K = M \times \mathfrak{g} \rightarrow M$$

and the vector bundle morphism $\Psi : K \rightarrow TM$ (over the identity of M) defined by

$$\Psi(x, \xi) = \xi_M(x), \quad (5.5)$$

where $\xi_M \in \mathfrak{X}(M)$ is the infinitesimal generator of the action ψ associated with $\xi \in \mathfrak{g}$.

A direct computation proves that the (unconstrained) momentum map $J_{(L, \Psi)} : E = TM \rightarrow K^* = M \times \mathfrak{g}^*$ associated with L and Ψ is given by

$$J_{(L, \Psi)}(v_x) = (x, J(v_x)), \text{ for } v_x \in T_x M,$$

where $J : TM \rightarrow \mathfrak{g}^*$ is the standard momentum map associated with L and the action ψ defined by

$$J(v_x)(\xi) = \frac{d}{dt} \Big|_{t=0} L(v_x + t\xi_M(x)), \text{ for } v_x \in T_x M \text{ and } \xi \in \mathfrak{g}$$

(see, for instance, [1]).

Now, each $\xi \in \mathfrak{g}$ defines a (constant) section σ of the vector bundle $K = M \times \mathfrak{g} \rightarrow M$ and the real function $J_{(L, \Psi)}^\sigma$ is just the momentum $J_\xi : TM \rightarrow \mathbb{R}$ in the direction of ξ .

On the other hand, if $\eta \in \mathfrak{g}$, then the infinitesimal generator η_{TM} of the tangent action $T\psi : G \times TM \rightarrow TM$ associated with η is the (standard) complete lift $\eta_M^\# \in \mathfrak{X}(TM)$ of η_M . Therefore, using Theorem 5.1, we deduce a well-known result [1]: “If the Lagrangian function $L : TM \rightarrow \mathbb{R}$ is invariant under the tangent action $T\psi$ of G on TM then, for every $\xi \in \mathfrak{g}$, the momentum $J_\xi : TM \rightarrow \mathbb{R}$ in the direction of ξ is a constant of the motion of the Lagrangian dynamics.” \bullet

5.2. Constrained case. Next, let us discuss the constrained case. Suppose that $L : E \rightarrow \mathbb{R}$ is a regular Lagrangian function on a Lie algebroid $\tau_E : E \rightarrow M$, that $\tau_K : K \rightarrow M$ is a vector bundle over M and that $\Psi : K \rightarrow E$ is a vector bundle morphism (over the identity of M) between K and E .

In addition, let $\tau_D : D \rightarrow M$ be a vector subbundle of $\tau_E : E \rightarrow M$ such that the nonholonomic Lagrangian system (L, D) is regular.

If x is point of M we consider the vector subspace K_x^D of K_x given by

$$K_x^D = \{k \in K_x / \Psi(k) \in D_x\}.$$

We will denote by $i_x : K_x^D \rightarrow K_x$ the canonical inclusion, by $i_x^* : K_x^* \rightarrow (K_x^D)^*$ the canonical projection and by K^D and $(K^D)^*$ the sets

$$K^D = \bigcup_{x \in M} K_x^D, \quad (K^D)^* = \bigcup_{x \in M} (K_x^D)^*.$$

Then, we define the **nonholonomic momentum map** $J_{(L, D, \Psi)} : E \rightarrow (K^D)^*$ **associated with the system (L, D) and the morphism Ψ** as follows

$$(J_{(L, D, \Psi)})|_{E_x} = i_x^* \circ (J_{(L, \Psi)})|_{E_x}, \quad \text{for } x \in M.$$

Now, if $\sigma : M \rightarrow K$ is a section of $\tau_K : K \rightarrow M$ such that $\sigma(x) \in K_x^D$, for all $x \in M$, we may introduce the real function $J_{(L, D, \Psi)}^\sigma : E \rightarrow \mathbb{R}$ given by

$$J_{(L, D, \Psi)}^\sigma(a) = J_{(L, D, \Psi)}(a)(\sigma(x)), \text{ for } a \in E_x,$$

that is, $J_{(L,D,\Psi)}^\sigma = J_{(L,\Psi)}^\sigma$.

Theorem 5.3 (The nonholonomic momentum equation). *Let $\Gamma_{(L,D)}$ be the solution of the constrained dynamics for the nonholonomic Lagrangian system (L,D) . If $\sigma : M \rightarrow K$ is a section of $\tau_K : K \rightarrow M$ such that $\sigma(x) \in K_x^D$, for all $x \in M$, and $(\Psi \circ \sigma)^c \in \text{Sec}(\mathcal{T}^E E)$ is the complete lift of $(\Psi \circ \sigma) \in \text{Sec}(E)$ then we have that*

$$\langle d^{T^E D}((J_{(L,D,\Psi)}^\sigma)|_D), \Gamma_{(L,D)} \rangle = \langle d^{T^E E} L, (\Psi \circ \sigma)^c \rangle|_D, \quad (5.6)$$

where $d^{T^E D}$ (respectively, $d^{T^E E}$) is the differential of Lie algebroid $\mathcal{T}^E D \rightarrow D$ (respectively, $\mathcal{T}^E E \rightarrow E$). In particular, if $\langle d^{T^E E} L, (\Psi \circ \sigma)^c \rangle|_D = 0$, then the real function $J_{(L,D,\Psi)}^\sigma$ is a constant of the motion for the constrained dynamics associated with the nonholonomic Lagrangian system (L,D) .

Proof. Denote by $j : D \rightarrow E$ and by $\mathcal{J} : \mathcal{T}^E D \rightarrow \mathcal{T}^E E$ the canonical inclusions and by $Q : \mathcal{T}_D^E E \rightarrow F$ the corresponding projector, where $F = \omega_L^{-1}(\tilde{D}^0)$ (see Section 3.3). Then, as we know,

$$\Gamma_{(L,D)} = (\Gamma_L - Q\Gamma_L)|_D.$$

Moreover, the pair (\mathcal{J}, j) is a Lie algebroid monomorphism which implies that

$$d^{T^E D}((J_{(L,D,\Psi)}^\sigma)|_D) = (\mathcal{J}, j)^*(d^{T^E E} J_{(L,D,\Psi)}^\sigma).$$

Thus, using that $J_{(L,D,\Psi)}^\sigma = J_{(L,\Psi)}^\sigma$ and proceedings as in the proof of Theorem 5.1, we deduce that

$$\begin{aligned} \langle d^{T^E D}((J_{(L,D,\Psi)}^\sigma)|_D), \Gamma_{(L,D)} \rangle &= \langle d^{T^E E} L, (\Psi \circ \sigma)^c \rangle|_D \\ &\quad - \{(\mathcal{L}_{(\Psi \circ \sigma)^c} \theta_L)(Q\Gamma_L) + (i(\Psi \circ \sigma)^c(\omega_L))(Q\Gamma_L)\}|_D. \end{aligned} \quad (5.7)$$

Now, since $S(Q\Gamma_L) = 0$, then $S[(\Psi \circ \sigma)^c, Q\Gamma_L] = 0$ (see (2.12)) and it follows that

$$\theta_L(Q\Gamma_L) = 0, \quad \theta_L[(\Psi \circ \sigma)^c, Q\Gamma_L] = 0.$$

Therefore,

$$(\mathcal{L}_{(\Psi \circ \sigma)^c} \theta_L)(Q\Gamma_L) = 0. \quad (5.8)$$

On the other hand, we have that

$$(i(Q\Gamma_L)\omega_L)|_D = S^*(\alpha_{(L,D)}), \text{ with } \alpha_{(L,D)} \in \text{Sec}((\mathcal{T}^E D)^0).$$

Consequently,

$$\{(i(\psi \circ \sigma)^c \omega_L)(Q\Gamma_L)\}|_D = -\alpha_{(L,D)}((\Psi \circ \sigma)|_D^v).$$

But, since $\Psi \circ \sigma$ is a section of $\tau_D : D \rightarrow M$, it follows that $(\Psi \circ \sigma)|_D^v$ is a section of $\mathcal{T}^E D \rightarrow D$. This implies that

$$\{(i(\Psi \circ \sigma)^c \omega_L)(Q\Gamma_L)\}|_D = 0. \quad (5.9)$$

Finally, using (5.7), (5.8) and (5.9), we conclude that (5.6) holds. \square

Remark 5.4 (Nonholonomic momentum equation on TM and horizontal symmetries). Suppose that $L : TM \rightarrow \mathbb{R}$ is an standard regular Lagrangian function on $E = TM$ and that $\psi : G \times M \rightarrow M$ is a (left) action of a Lie group G on M . Then, we consider the trivial vector bundle $\tau_K : K = M \times \mathfrak{g} \rightarrow M$ and the vector bundle morphism $\Psi : K \rightarrow TM$ (over the identity of M) defined by (5.5).

Now, let D be a vector subbundle (over M) of the vector bundle $\tau_M : TM \rightarrow M$, that is, D is a distribution on M , and assume that the nonholonomic Lagrangian system (L,D) is regular. If x is a point of M , we have that $K_x^D = \{x\} \times \mathfrak{g}^x$, where \mathfrak{g}^x is the vector subspace of \mathfrak{g} given by

$$\mathfrak{g}^x = \{\xi \in \mathfrak{g} / \xi_M(x) \in D_x\}.$$

We also remark that the sets K^D and $(K^D)^*$ may be identified with the sets

$$\mathfrak{g}^D = \bigcup_{x \in M} \mathfrak{g}^x, \quad (\mathfrak{g}^D)^* = \bigcup_{x \in M} (\mathfrak{g}^x)^*.$$

Under this identification, the nonholonomic momentum map $J_{(L,D,\Psi)} : E \rightarrow (K^D)^*$ associated with the system (L, D) and the morphism Ψ is just the standard nonholonomic momentum map $J^{nh} : TM \rightarrow (\mathfrak{g}^D)^*$ associated with the system (L, D) and the action ψ (see [4, 8, 9]).

Now, if $\tilde{\xi} : M \rightarrow \mathfrak{g}$ is a smooth map the $\tilde{\xi}$ defines, in a natural way, a section $\sigma_{\tilde{\xi}} : M \rightarrow K = M \times \mathfrak{g}$ of the vector bundle $\tau_K : K = M \times \mathfrak{g} \rightarrow M$. We denote by $J_{\tilde{\xi}}^{nh} : TM \rightarrow \mathbb{R}$ the real function $J_{(L,D,\Psi)}^{\sigma_{\tilde{\xi}}} : E \rightarrow \mathbb{R}$ and by $\Xi_{\tilde{\xi}}$ the vector field $\Psi \circ \sigma_{\tilde{\xi}}$ on M . Then, using Theorem 5.3, we deduce a well-known result (see [4, 8, 9]): “If $\Gamma_{(L,D)}$ is the solution of the constrained dynamics for the nonholonomic system (L, D) , we have that

$$\Gamma_{L,D}((J_{\tilde{\xi}}^{nh})|_D) = (\Xi_{\tilde{\xi}}|_D)^c(L).”$$

The above equality is an intrinsic expression of the **standard nonholonomic momentum equation**. In addition, using again Theorem 5.3 we also deduce another well-known result (see [4, 8, 9]): “If the Lagrangian function $L : TM \rightarrow \mathbb{R}$ is invariant under the tangent action $T\psi$ of G on TM and $\xi \in \mathfrak{g}$ is a **horizontal symmetry** (that is, $\xi \in \mathfrak{g}^x$, for all $x \in M$) then the real function $(J_{\tilde{\xi}}^{nh})|_D$ is a constant of the motion for the constrained Lagrangian dynamics, where $\tilde{\xi} : M \rightarrow \mathfrak{g}$ is the constant map

$$\tilde{\xi}(x) = \xi, \text{ for all } x \in M.” \quad \bullet$$

6. EXAMPLES

As in the unconstrained case, constrained Lagrangian systems on Lie algebroids appear frequently. We show some examples next.

6.1. Nonholonomic Lagrangian systems on Lie algebras. Let \mathfrak{g} be a real algebra of finite dimension. Then, it is clear that \mathfrak{g} is a Lie algebroid over a single point. Now, suppose that (l, \mathfrak{d}) is a nonholonomic Lagrangian system on \mathfrak{g} , that is, $l : \mathfrak{g} \rightarrow \mathbb{R}$ is a Lagrangian function and \mathfrak{d} is a vector subspace of \mathfrak{g} . If $w : I \rightarrow \mathfrak{g}$ is a curve on \mathfrak{g} then

$$dl(\omega(t)) \in T_{\omega(t)}^* \mathfrak{g} \cong \mathfrak{g}^*, \quad \forall t \in I,$$

and thus, the map $dl \circ \omega$ may be considered as a curve on \mathfrak{g}^*

$$dl \circ \omega : I \rightarrow \mathfrak{g}^*.$$

Therefore,

$$(dl \circ \omega)'(t) \in T_{dl(\omega(t))} \mathfrak{g}^* \cong \mathfrak{g}^*, \quad \forall t \in I.$$

Moreover, from (3.7), it follows that ω is a solution of the Lagrange-d'Alembert equations for the system (l, \mathfrak{d}) if and only if

$$(dl \circ \omega)'(t) - ad_{\omega(t)}^*(dl(\omega(t))) \in \mathfrak{d}^\circ, \quad \omega(t) \in \mathfrak{d}, \quad \forall t \quad (6.1)$$

where $ad^* : \mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is the infinitesimal coadjoint action.

The above equations are just the so-called **Euler-Poincaré -Suslov equations** for the system (l, \mathfrak{d}) (see [28]). We remark that in the particular case when the system is unconstrained, that is, $\mathfrak{d} = \mathfrak{g}$, then one recovers the **the standard Euler-Poincaré equations** for the Lagrangian function $l : \mathfrak{g} \rightarrow \mathbb{R}$.

If G is a Lie group with Lie algebra \mathfrak{g} then nonholonomic Lagrangian systems on \mathfrak{g} may be obtained (by reduction) from nonholonomic LL mechanical systems with configuration space the Lie group G .

In fact, let e be the identity element of G and $\mathbb{I} : \mathfrak{g} \rightarrow \mathfrak{g}^*$ be a symmetric positive definite inertia operator. Denote by $g_e : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ the corresponding scalar product on \mathfrak{g} given by

$$g_e(\omega, \omega') = \langle \mathbb{I}(\omega), \omega' \rangle, \text{ for } \omega, \omega' \in \mathfrak{g} \cong T_e G.$$

g_e induces a left-invariant Riemannian metric g on G . Thus, we may consider the Lagrangian function $L : TG \rightarrow \mathbb{R}$ defined by

$$L(v_h) = \frac{1}{2} g_h(v_h, v_h), \quad \text{for } v_h \in T_h G.$$

In other words, L is the kinetic energy associated with the Riemannian metric g .

Now, let D be a left-invariant distribution on G . Then, since L is a left-invariant function, the pair (L, D) is an standard nonholonomic LL system in the terminology of [28].

On the other hand, the Lagrangian momentum map $\Phi : TG \rightarrow \mathfrak{g}$ given by

$$\Phi(v_h) = (T_h l_{h^{-1}})(v_h), \text{ for } v_h \in T_h G$$

is a fiberwise bijective morphism of Lie algebroids. Moreover, if $l = L|_{\mathfrak{g}}$ and $\mathfrak{d} = D_e$ then the pair (l, \mathfrak{d}) is a nonholonomic Lagrangian system on \mathfrak{g} and

$$l \circ \Phi = L \text{ and } \Phi(D) = \mathfrak{d}.$$

Thus, the system (l, \mathfrak{d}) is regular. In addition, if $v : I \rightarrow TG$ is a solution of the Lagrange-d'Alembert equations for the system (L, D) then, using Theorem 4.6, we deduce that the curve $\Phi \circ v : I \rightarrow \mathfrak{g}$ is a solution of the Lagrange-d'Alembert equations for the system (l, \mathfrak{d}) .

We remark that

$$l(\omega) = \frac{1}{2} g_e(\omega, \omega) = \frac{1}{2} \langle \mathbb{I}(\omega), \omega \rangle, \text{ for } \omega \in \mathfrak{g}.$$

Therefore, if $\omega : I \rightarrow \mathfrak{g}$ is a curve on \mathfrak{g} , we have that

$$(dl \circ \omega)(t) = \mathbb{I}(\omega(t)), \text{ for all } t$$

and, using (6.1), it follows that ω is a solution of the Lagrange-d'Alembert equations for the system (l, \mathfrak{d}) if and only if

$$\dot{\omega} - \mathbb{I}^{-1}(ad_{\omega(t)}^* \mathbb{I}(\omega(t))) \in \mathfrak{d}^\perp, \quad \omega(t) \in \mathfrak{d}, \text{ for all } t,$$

where \mathfrak{d}^\perp is the orthogonal complement of the subspace \mathfrak{d} , that is,

$$\mathfrak{d}^\perp = \{\omega' \in \mathfrak{g} / \langle \mathbb{I}(\omega'), \omega \rangle = 0, \forall \omega \in \mathfrak{d}\}.$$

Two simple examples of the above general situation are the following ones.

The Suslov system. The most natural example of LL system is the *nonholonomic Suslov problem*, which describes the motion of a rigid body about a fixed point under the action of the following nonholonomic constraint: the body angular velocity vector is orthogonal to a some fixed direction in the body frame.

The configuration space of the problem is the group $G = SO(3)$. Thus, in this case, the Lie algebra \mathfrak{g} may be identified with \mathbb{R}^3 and, under this identification, the Lie bracket on \mathfrak{g} is just the cross product \times on \mathbb{R}^3 .

Moreover, if $\mathbb{I} : \mathbb{R}^3 \rightarrow (\mathbb{R}^3)^* \cong \mathbb{R}^3$ is the inertia tensor of the body then a curve $\omega : I \rightarrow \mathbb{R}^3$ on \mathbb{R}^3 is a solution of the Euler-Poincaré-Suslov equations for the system if and only if

$$\dot{\omega} = \mathbb{I}^{-1}((\mathbb{I}\omega) \times \omega) + \lambda \mathbb{I}^{-1}(\Gamma), \quad \langle \omega, \Gamma \rangle = 0, \quad (6.2)$$

where λ is the Lagrange multiplier, Γ is a fixed unit vector in \mathbb{R}^3 and $\langle \cdot, \cdot \rangle$ is the standard scalar product in \mathbb{R}^3 . Since the nonholonomic system is regular, the

Lagrange multiplier λ is uniquely determined. In fact, differentiating the equation $\langle \omega, \Gamma \rangle = 0$, we find

$$\lambda = -\frac{\langle \mathbb{I}\omega \times \omega, \mathbb{I}^{-1}\Gamma \rangle}{\langle \Gamma, \mathbb{I}^{-1}\Gamma \rangle}$$

and, consequently, Eqs. (6.2) are equivalent to

$$\dot{\omega} = \mathbb{I}^{-1}(\langle \mathbb{I}\omega, \Gamma \rangle \omega \times \mathbb{I}^{-1}\Gamma), \quad \langle \omega, \Gamma \rangle = 0.$$

Multidimensional generalizations of the Suslov problem have been discussed by several authors (see [27, 34, 69]).

The Chaplygin sleigh. The Chaplygin sleigh is a rigid body sliding on a horizontal plane. The body is supported at three points, two of which slide freely without friction while the third is a knife edge, a constraint that allows no motion orthogonal to this edge. This mechanical system was introduced and studied in 1911 by Chaplygin [16] (see also [57]).

The configuration space of this system is the group $SE(2)$ of Euclidean motions of the two-dimensional plane \mathbb{R}^2 . As we know, we may choose local coordinates (θ, x, y) on $SE(2)$. θ and (x, y) are the angular orientation of the blade and position of the contact point of the blade on the plane, respectively.

Now, we introduce a coordinate system called the body frame by placing the origin at the contact point and choosing the first coordinate axis in the direction of the knife edge. Denote the angular velocity of the body by $\omega = \dot{\theta}$ and the components of the linear velocity of the contact point relative to the body frame by v_1, v_2 . The set (ω, v_1, v_2) is regarded as an element of the Lie algebra $\mathfrak{se}(2)$. Note that

$$v_1 = \dot{x} \cos \theta + \dot{y} \sin \theta, \quad v_2 = \dot{y} \cos \theta - \dot{x} \sin \theta.$$

The position of the center of mass is specified by the coordinates (a, b) relative to the body frame. Let m and J denote the mass and moment of inertia of the sleigh relative to the contact point. Then, the corresponding symmetric positive definite inertia operator $\mathbb{I} : \mathfrak{se}(2) \rightarrow \mathfrak{se}(2)^*$ and the reduced nonholonomic Lagrangian system (l, \mathfrak{d}) on $\mathfrak{se}(2)$ are given by

$$\begin{aligned} \mathbb{I}(\omega, v_1, v_2) &= \begin{pmatrix} J + m(a^2 + b^2) & -bm & am \\ -bm & m & 0 \\ am & 0 & m \end{pmatrix} \begin{pmatrix} \omega \\ v_1 \\ v_2 \end{pmatrix}, \\ l(\omega, v_1, v_2) &= \frac{1}{2}[(J + m(a^2 + b^2))\omega^2 + m(v_1^2 + v_2^2) \\ &\quad - 2mb\omega v_1 + 2am\omega v_2], \\ \mathfrak{d} &= \{(\omega, v_1, v_2) \in \mathfrak{se}(2) / v_2 = 0\}, \end{aligned}$$

(see [28]). Thus, the Lagrange-d'Alembert equations for the system (l, \mathfrak{d}) are

$$\begin{aligned} \dot{\omega} &= \frac{am\omega}{J + ma^2}(b\omega - v_1), \\ \dot{v}_1 &= \frac{a\omega}{J + ma^2}((J + m(a^2 + b^2))\omega - mbv_1), \\ \dot{v}_2 &= 0. \end{aligned}$$

Multidimensional generalizations of the Chaplygin sleigh were discussed in [28] (see also [57] and [70]).

6.2. Nonholonomic LR systems and right action Lie algebroids. Here, we show how the reduction of a nonholonomic LR system produces a nonholonomic Lagrangian system on a right action Lie algebroid.

Let us start by recalling the definition of a right action Lie algebroid (see [32]). Let $(F, [\cdot, \cdot]_F, \rho_F)$ be a Lie algebroid over a manifold N and $\pi : M \rightarrow N$ be a smooth map. **A right action of F on $\pi : M \rightarrow N$** is a \mathbb{R} -linear map

$$\Psi : \text{Sec}(F) \rightarrow \mathfrak{X}(M), \quad X \in \text{Sec}(F) \rightarrow \Psi(X) \in \mathfrak{X}(M)$$

such that

$$\begin{aligned} \Psi(fX) &= (f \circ \pi)\Psi(X), \quad \Psi([X, Y]_F) = [\Psi(X), \Psi(Y)], \\ (T_m\pi)(\Psi(X)(m)) &= \rho_F(X(\pi(m))), \end{aligned}$$

for $f \in C^\infty(N)$, $X, Y \in \text{Sec}(F)$ and $m \in M$. If $\Psi : \text{Sec}(F) \rightarrow \mathfrak{X}(M)$ is a right action of F on $\pi : M \rightarrow N$ and $\tau_F : F \rightarrow N$ is the vector bundle projection then the pullback vector bundle of F over π ,

$$E = \pi^*F = \{(m, f) \in M \times F / \tau_F(f) = \pi(m)\}$$

is a Lie algebroid over M with Lie algebroid structure $([\cdot, \cdot]_E, \rho_E)$ which is characterized by

$$[X, Y]_E = [X, Y]_F \circ \pi, \quad \rho_E(X)(m) = \Psi(X)(m),$$

for $X, Y \in \text{Sec}(E)$ and $m \in M$. The triple $(E, [\cdot, \cdot]_E, \rho_E)$ is called the **right action Lie algebroid of F over π** and it is denoted by $\pi_\Psi F$ (see [32]).

Note that if the Lie algebroid F is a real Lie algebra \mathfrak{g} of finite dimension and $\pi : M \rightarrow \{\text{a point}\}$ is the constant map then a right action of \mathfrak{g} on π is just a right infinitesimal action $\Psi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ of \mathfrak{g} on the manifold M . In such a case, the corresponding right action Lie algebroid is the trivial vector bundle $pr_1 : M \times \mathfrak{g} \rightarrow M$.

Next we recall the definition of a nonholonomic LR system following [26, 35]. Let G be a compact connected Lie group with Lie algebra \mathfrak{g} and $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ be an Ad_G -invariant scalar product on \mathfrak{g} . Now, suppose that $\mathcal{I} : \mathfrak{g} \rightarrow \mathfrak{g}$ is a inertia operator which is symmetric and definite positive with respect to the scalar product $\langle \cdot, \cdot \rangle$. Denote by g the left-invariant Riemannian metric given by

$$g_h(v_h, v_h') = \langle \mathcal{I}(T_h l_{h^{-1}}(v_h)), (T_h l_{h^{-1}})(v_h') \rangle \quad (6.3)$$

for $h \in G$ and $v_h, v_h' \in T_h G$.

Then, the Lagrangian function $L : TG \rightarrow \mathbb{R}$ of the system is

$$L(v_h) = \frac{1}{2}g_h(v_h, v_h) - V(h), \quad \text{for } v_h \in T_h G, \quad (6.4)$$

$V : G \rightarrow \mathbb{R}$ being the potential energy. The constraint distribution D is a right-invariant distribution on G . Thus, if e is the identity element of G and $\mathfrak{d} = D_e$, we have that

$$D_h = (T_e r_h)(\mathfrak{d}) = (T_e l_h)(Ad_{h^{-1}}(\mathfrak{d})), \quad \text{for } h \in G \quad (6.5)$$

where $Ad : G \times \mathfrak{g} \rightarrow \mathfrak{g}$ is the adjoint action.

The nonholonomic Lagrangian system (L, D) on TG is called a **nonholonomic LR system** in the terminology of [26, 35]. Note that, since L is a Lagrangian function of mechanical type, the system (L, D) is regular. Now, assume that

$$\mathfrak{s} = \mathfrak{d}^\perp = \{\omega' \in \mathfrak{g} / \langle \omega, \omega' \rangle = 0, \forall \omega \in \mathfrak{d}\}$$

is a Lie subalgebra of \mathfrak{g} , that S is a closed Lie subgroup of G with Lie algebra \mathfrak{s} and that the potential energy V is S -invariant.

Next, let us show that the nonholonomic LR system (L, D) may be reduced to a nonholonomic Lagrangian system on a right action Lie algebroid. In fact, consider the Riemannian homogeneous space $M = S \backslash G$ and the standard transitive right

action ψ of G on $M = S \setminus G$. Denote by $\Psi : \mathfrak{g} \rightarrow \mathfrak{X}(S \setminus G)$ the corresponding right infinitesimal action of \mathfrak{g} on $S \setminus G$. Then, Ψ induces a Lie algebroid structure on the trivial vector bundle $pr_1 : S \setminus G \times \mathfrak{g} \rightarrow S \setminus G$.

On the other hand, using that the potential energy V is S invariant, we deduce that V induces a real function $\tilde{V} : S \setminus G \rightarrow \mathbb{R}$ on $S \setminus G$ such that

$$\tilde{V} \circ \pi = V, \quad (6.6)$$

where $\pi : G \rightarrow S \setminus G$ is the canonical projection. Thus, we can introduce the Lagrangian function $\tilde{L} : S \setminus G \times \mathfrak{g} \rightarrow \mathbb{R}$ on the action Lie algebroid $pr_1 : S \setminus G \times \mathfrak{g} \rightarrow S \setminus G$ defined by

$$\tilde{L}(\tilde{h}, \omega) = \frac{1}{2} \langle \mathcal{I}(\omega), \omega \rangle - \tilde{V}(\tilde{h}), \quad \text{for } \tilde{h} \in S \setminus G \text{ and } \omega \in \mathfrak{g}. \quad (6.7)$$

Now, for every $h \in G$, we consider the subspace $\mathfrak{d}(h)$ of \mathfrak{g} given by

$$\mathfrak{d}(h) = Ad_{h^{-1}}(\mathfrak{d}). \quad (6.8)$$

The dimension of $\mathfrak{d}(h)$ is equal to the dimension of \mathfrak{d} . Moreover, since $\langle \cdot, \cdot \rangle$ is Ad_G -invariant, it follows that

$$\mathfrak{d}(h) = (Ad_{h^{-1}}(\mathfrak{s}))^\perp = \{\omega' \in \mathfrak{g} / \langle \omega', Ad_{h^{-1}}(\omega) \rangle = 0, \forall \omega \in \mathfrak{s}\}.$$

In particular, we have that

$$\mathfrak{d}(s) = \mathfrak{s}^\perp = \mathfrak{d}, \quad \forall s \in S$$

which implies that $\mathfrak{d}(sh) = \mathfrak{d}(h)$, for all $h \in G$.

Therefore, we can define a vector subbundle D of the Lie algebroid $pr_1 : S \setminus G \times \mathfrak{g} \rightarrow S \setminus G$ as follows

$$\tilde{D}_{\tilde{h}} = \{\tilde{h}\} \times \mathfrak{d}(h), \quad \text{for } \tilde{h} \in S \setminus G \quad (6.9)$$

with $h \in G$ and $\pi(h) = \tilde{h}$. Consequently, the pair (\tilde{L}, \tilde{D}) is a nonholonomic Lagrangian system on the action Lie algebroid $pr_1 : S \setminus G \times \mathfrak{g} \rightarrow S \setminus G$.

In addition, we may prove the following result

Proposition 6.1. (1) *If $\tilde{\Phi} : TG \rightarrow S \setminus G \times \mathfrak{g}$ is the map given by*

$$\tilde{\Phi}(v_h) = (\pi(h), (T_h l_{h^{-1}}(v_h))), \quad \text{for all } v_h \in T_h G \quad (6.10)$$

then $\tilde{\Phi}$ is a fiberwise bijective Lie algebroid morphism over π .

(2) *The nonholonomic Lagrangian systems (L, D) and (\tilde{L}, \tilde{D}) on TG and $S/G \times \mathfrak{g}$ are $\tilde{\Phi}$ -related, that is,*

$$\tilde{L} \circ \tilde{\Phi} = L, \quad \tilde{\Phi}(D) = \tilde{D}.$$

(3) *The system (\tilde{L}, \tilde{D}) is regular and if $\gamma : I \rightarrow TG$ is a solution of the Lagrange-d'Alembert equations for the system (L, D) then $\tilde{\Phi} \circ \gamma : I \rightarrow S \setminus G \times \mathfrak{g}$ is a solution of the Lagrange-d'Alembert equations for the system (\tilde{L}, \tilde{D}) .*

Proof. (1) Consider the standard (right) action r of G on itself

$$r : G \times G \rightarrow G, \quad (h, h') \in G \times G \rightarrow r_{h'}(h) = hh' \in G.$$

As we know, the infinitesimal generator of r associated with an element ω of \mathfrak{g} is

$$\omega_G = \overleftarrow{\omega},$$

where $\overleftarrow{\omega}$ is the left-invariant vector field on G such that $\overleftarrow{\omega}(e) = \omega$.

On the other hand, it is clear that the projection $\pi : G \rightarrow S \setminus G$ is equivariant with respect to the actions r and ψ . Thus,

$$(T_h \pi)(\overleftarrow{\omega}(h)) = \Psi(\omega)(\pi(h)), \quad \text{for } h \in G.$$

Therefore, if $\rho : S/G \times \mathfrak{g} \rightarrow T(S \setminus G)$ is the anchor map of the Lie algebroid $pr_1 : S \setminus G \times \mathfrak{g} \rightarrow S \setminus G$, it follows that

$$\rho(\tilde{\Phi}(\overleftarrow{\omega}(h))) = \rho(\pi(h), \omega) = (T_h \pi)(\overrightarrow{\omega}(h)), \quad \text{for } h \in G.$$

Furthermore, since

$$[\overleftarrow{\omega}, \overleftarrow{\omega'}] = \overleftarrow{[\omega, \omega']}_{\mathfrak{g}}, \quad \text{for } \omega, \omega' \in \mathfrak{g},$$

we conclude that $\tilde{\Phi}$ is a Lie algebroid morphism over π .

In addition, it is obvious that if $h \in G$ then

$$\tilde{\Phi}|_{T_h G} : T_h G \rightarrow \{\pi(h)\} \times \mathfrak{g}$$

is a linear isomorphism.

(2) From (6.3), (6.4), (6.6), (6.7) and (6.10), we deduce that

$$\tilde{L} \circ \tilde{\Phi} = L.$$

Moreover, using (6.5), (6.8), (6.9) and (6.10), we obtain that

$$\tilde{\Phi}(D) = \tilde{D}.$$

(3) It follows from (1), (2) and using the results of Section 4 (see Theorem 4.6). \square

Next, we obtain the necessary and sufficient conditions for a curve $(\tilde{h}, \omega) : I \rightarrow S \setminus G \times \mathfrak{g}$ to be a solution of the Lagrange-d'Alembert equations for the system (\tilde{L}, \tilde{D}) . Let $\flat_{<\cdot, \cdot>} : \mathfrak{g} \rightarrow \mathfrak{g}^*$ be the linear isomorphism induced by the scalar product $<\cdot, \cdot> : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ and $\mathbb{I} : \mathfrak{g} \rightarrow \mathfrak{g}^*$ be the inertia operator given by

$$\mathbb{I}(w_1)(w_2) = \langle \mathcal{I}(\omega_1), \omega_2 \rangle, \quad \text{for } \omega_1, \omega_2 \in \mathfrak{g}. \quad (6.11)$$

On the other hand, if $\tilde{h}' \in S \setminus G$ we will denote by $\Psi_{\tilde{h}'} : \mathfrak{g} \rightarrow T_{\tilde{h}'}(S \setminus G)$ the linear epimorphism defined by

$$\Psi_{\tilde{h}'}(\omega') = \Psi(\omega')(\tilde{h}'), \quad \text{for } \omega' \in \mathfrak{g}.$$

In addition, if $\pi(h') = \tilde{h}'$, we identify the vector space $\tilde{D}_{\tilde{h}'}$ with the vector subspace $\mathfrak{d}(h')$ of \mathfrak{g} . Then, using (3.7), (6.7) and (6.11), we deduce that the curve (\tilde{h}, ω) is a solution of the Lagrange-d'Alembert equations for the system (\tilde{L}, \tilde{D}) if and only if

$$\begin{aligned} \dot{\tilde{h}}(t) &= \Psi_{\tilde{h}(t)}(\omega(t)) \\ \{\dot{\omega}(t) - \mathbb{I}^{-1}(ad_{\omega(t)}^* \mathbb{I}(\omega(t))) - \mathbb{I}^{-1}(\Psi_{\tilde{h}(t)}^*(d\tilde{V}(\tilde{h}(t))))\} &\in \tilde{D}_{\tilde{h}(t)}^\perp, \\ \omega(t) &\in \tilde{D}_{\tilde{h}(t)}, \end{aligned}$$

for all t , where $\tilde{D}_{\tilde{h}(t)}^\perp$ is the orthogonal complement of the vector subspace $\tilde{D}_{\tilde{h}(t)} \subseteq \mathfrak{g}$ with respect to the scalar product $<\cdot, \cdot>$. These equations will be called **the reduced Poincaré-Chetaev equations**.

We treat next a simple example of the above general situation.

The Veselova system. The most descriptive illustration of an LR system is the Veselova problem on the motion of a rigid body about a fixed point under the action of the nonholonomic constraint

$$<\omega, \gamma> = 0.$$

Here, ω is the vector of the angular velocity in the body frame, γ is a unit vector which is fixed in an space frame and $<\cdot, \cdot>$ denotes the standard scalar product in \mathbb{R}^3 (see [66]).

The Veselova system is an LR system on the Lie group $G = SO(3)$ which is the configuration space of the rigid body motion. Thus, in this case, the Lie algebra \mathfrak{g}

may be identified with \mathbb{R}^3 and, under this identification, the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$ is the cross product \times on \mathbb{R}^3 . Moreover, the adjoint action of $G = SO(3)$ on $\mathfrak{g} \cong \mathbb{R}^3$ is the standard action of $SO(3)$ on \mathbb{R}^3 . This implies that $\langle \cdot, \cdot \rangle$ is an $Ad_{SO(3)}$ -invariant scalar product on $\mathfrak{g} \cong \mathbb{R}^3$.

The vector subspace \mathfrak{d} of \mathbb{R}^3 is just the orthogonal complement (with respect to $\langle \cdot, \cdot \rangle$) of a vector subspace $\langle \gamma_0 \rangle$ of dimension 1, with γ_0 a unit vector in \mathbb{R}^3 , that is,

$$\mathfrak{d} = \{\omega \in \mathbb{R}^3 / \langle \omega, \gamma_0 \rangle = 0\}.$$

Therefore,

$$\mathfrak{s} = \mathfrak{d}^\perp = \langle \gamma_0 \rangle$$

is a Lie subalgebra of $\mathfrak{g} \cong \mathbb{R}^3$. Furthermore, the isotropy group S of γ_0 with respect to the adjoint action of $G = SO(3)$,

$$S = \{s \in SO(3) / s\gamma_0^T = \gamma_0^T\},$$

is a closed Lie subgroup with Lie algebra \mathfrak{s} . We remark that S is isomorphic to the circle S^1 .

Consequently, the corresponding homogeneous space $M = S \backslash SO(3)$ is the orbit of the adjoint action of $SO(3)$ on \mathbb{R}^3 over the point γ_0 and, it is well-known that, such an orbit may be identified with the unit sphere S^2 . In fact, the map

$$S \backslash SO(3) \rightarrow S^2, \quad [h] \rightarrow \gamma_0 h = (h^{-1} \gamma_0^T)^T$$

is a diffeomorphism (see, for instance, [50]).

Under the above identification the (right) action of $SO(3)$ on $M = S \backslash SO(3)$ is just the standard (right) action of $SO(3)$ on S^2 . Thus, our action Lie algebroid is the trivial vector bundle $pr_1 : S^2 \times \mathbb{R}^3 \rightarrow S^2$ and the Lie algebroid structure on it is induced by the standard infinitesimal (right) action $\Psi : \mathbb{R}^3 \rightarrow \mathfrak{X}(S^2)$ of the Lie algebra (\mathbb{R}^3, \times) on S^2 defined by

$$\Psi(\omega)(\gamma) = \gamma \times \omega, \quad \text{for } \omega \in \mathbb{R}^3 \text{ and } \gamma \in S^2.$$

In the presence of a potential $\tilde{V} : \gamma \rightarrow \tilde{V}(\gamma)$ the nonholonomic Lagrangian system (\tilde{L}, \tilde{D}) on the Lie algebroid $pr_1 : S^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is given by

$$\tilde{L}(\gamma, \omega) = \frac{1}{2} \mathbb{I}(\omega)(\omega) - \tilde{V}(\gamma), \quad \tilde{D}(\gamma) = \{\gamma\} \times \{\omega \in \mathbb{R}^3 / \langle \omega, \gamma \rangle = 0\},$$

$\mathbb{I} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ being the inertia tensor of the rigid body.

The Lagrange-d'Alembert equations for (\tilde{L}, \tilde{D}) are

$$\dot{\gamma} = \gamma \times \omega, \quad \dot{\omega} = \mathbb{I}^{-1} \left\{ (\mathbb{I}\omega \times \omega) + \omega \times \frac{\partial \tilde{V}}{\partial \gamma} + \lambda \gamma \right\}, \quad \langle \omega, \gamma \rangle = 0 \quad (6.12)$$

where λ is the Lagrange multiplier. Since the system (\tilde{L}, \tilde{D}) is regular, λ is uniquely determined. In fact,

$$\lambda = - \frac{\langle \mathbb{I}\omega \times \omega + \gamma \times \frac{\partial \tilde{V}}{\partial \gamma}, \mathbb{I}^{-1} \gamma \rangle}{\langle \mathbb{I}^{-1} \gamma, \gamma \rangle}. \quad (6.13)$$

Eqs (6.12) and (6.13) are just the classical dynamical equations for the Veselova system (see [66]; see also [26, 35]).

6.3. Semidirect product symmetry and left action Lie algebroids. Here, we show how the reduction of some nonholonomic mechanical systems with semidirect product symmetry produces nonholonomic Lagrangian systems on left action Lie algebroids.

Let us start by recalling the definition of a left action Lie algebroid (see [32]). Let $(F, [\cdot, \cdot]_F, \rho_F)$ be a Lie algebroid over a manifold N and $\pi : M \rightarrow N$ be a smooth map. **A left action of F on $\pi : M \rightarrow N$** is a \mathbb{R} -linear map

$$\Psi : \text{Sec}(F) \rightarrow \mathfrak{X}(M), \quad X \in \text{Sec}(F) \rightarrow \Psi(X) \in \mathfrak{X}(M)$$

such that

$$\begin{aligned} \Psi(fX) &= (f \circ \pi)\Psi(X), \quad \Psi([X, Y]_F) = -[\Psi(X), \Psi(Y)], \\ (T_m\pi)(\Psi(X)(m)) &= -\rho_F(X(\pi(m))), \end{aligned}$$

for $f \in C^\infty(N)$, $X, Y \in \text{Sec}(F)$ and $m \in M$. If $\Psi : \text{Sec}(F) \rightarrow \mathfrak{X}(M)$ is a left action of F on $\pi : M \rightarrow N$ and $\tau_F : F \rightarrow N$ is the vector bundle projection then the pullback vector bundle of F over π ,

$$E = F^*\pi = \{(f, m) \in F \times M / \tau_F(f) = \pi(m)\}$$

is a Lie algebroid over M with Lie algebroid structure $([\cdot, \cdot]_E, \rho_E)$ which is characterized by

$$[X, Y]_E = [X, Y]_F \circ \pi, \quad \rho_E(X)(m) = -\Psi(X)(m),$$

for $X, Y \in \text{Sec}(E)$ and $m \in M$. The triple $(E, [\cdot, \cdot]_E, \rho_E)$ is called **the left action Lie algebroid of F over π** and it is denoted by $F_\Psi\pi$ (see [32]).

Next, we consider a particular class of nonholonomic Lagrangian systems on left action Lie algebroids. Let V be a real vector space of finite dimension and $\cdot : G \times V \rightarrow V$ be a left representation of a Lie group G on V . We also denote by $\cdot : \mathfrak{g} \times V \rightarrow V$ the left infinitesimal representation of the Lie algebra \mathfrak{g} of G on V . Then, we can consider the semidirect Lie group $S = G \ltimes V$ with the multiplication

$$(g, v)(g', v') = (gg', v + g \cdot v').$$

The Lie algebra \mathfrak{s} of S is the semidirect product $\mathfrak{s} = \mathfrak{g} \ltimes V$ with the Lie bracket $[\cdot, \cdot]_{\mathfrak{s}} : \mathfrak{s} \times \mathfrak{s} \rightarrow \mathfrak{s}$ given by

$$[(\omega, \dot{v}), (\omega', \dot{v}')]_{\mathfrak{s}} = ([\omega, \omega']_{\mathfrak{g}}, \omega \cdot \dot{v}' - \omega' \cdot \dot{v})$$

for $\omega, \omega' \in \mathfrak{g}$ and $\dot{v}, \dot{v}' \in V$. Here, $[\cdot, \cdot]_{\mathfrak{g}}$ is the Lie bracket on \mathfrak{g} .

Moreover, we use the following notation. If $v \in V$ then $\rho_v : \mathfrak{g} \rightarrow V$ is the linear map defined by

$$\rho_v(\omega) = \omega \cdot v, \quad \text{for } \omega \in \mathfrak{g},$$

and $\rho_v^* : V^* \rightarrow \mathfrak{g}^*$ is the dual map of $\rho_v : \mathfrak{g} \rightarrow V$.

Now, let N be a smooth manifold. Then, it is clear that the product manifold $F = \mathfrak{s} \times TN$ is the total space of a vector bundle over N . Moreover, if $(\omega, \dot{v}) \in \mathfrak{s}$ and X is a vector field on N then the pair $((\omega, \dot{v}), X)$ defines a section of the vector bundle $\tau_F : F = \mathfrak{s} \times TN \rightarrow N$. In fact, if $\{\omega_i\}$ is a basis of \mathfrak{g} , $\{\dot{v}_j\}$ is a basis of V and $\{X_k\}$ is a local basis of $\mathfrak{X}(N)$ then $\{((\omega_i, 0), 0), ((0, \dot{v}_j), 0), ((0, 0), X_k)\}$ is a local basis of $\text{Sec}(F)$.

The vector bundle $\tau_F : F \rightarrow N$ admits a Lie algebroid structure $([\cdot, \cdot]_F, \rho_F)$, which is characterized by the following relations

$$\begin{aligned} [((\omega, \dot{v}), X), ((\omega', \dot{v}'), X')]_F &= ([(\omega, \dot{v}), (\omega', \dot{v}')]_{\mathfrak{s}}, [X, X']) \\ &= ([\omega, \omega']_{\mathfrak{g}}, \omega \cdot \dot{v}' - \omega' \cdot \dot{v}, [X, X']), \\ \rho_F((\omega, \dot{v}), X) &= X, \end{aligned} \tag{6.14}$$

for $((\omega, \dot{v}), X), ((\omega', \dot{v}'), X') \in \mathfrak{s} \times \mathfrak{X}(N)$.

Next, suppose that v_0 is a point of V and that \mathcal{O}_{v_0} is the orbit of the action of G on V by v_0 , that is,

$$\mathcal{O}_{v_0} = \{g \cdot v_0 \in V / g \in G\}.$$

Denote by $\pi : M = N \times \mathcal{O}_{v_0} \rightarrow N$ the canonical projection on the first factor and by $\Psi : \text{Sec}(F) \rightarrow \mathfrak{X}(M)$ the left action of F on π , which is characterized by the following relation

$$\Psi((\omega, \dot{u}), X)(n, v) = (-X(n), \omega \cdot v)$$

for $((\omega, \dot{u}), X) \in \mathfrak{s} \times \mathfrak{X}(N)$ and $(n, v) \in N \times \mathcal{O}_{v_0} = M$.

Then, we have the corresponding left action Lie algebroid $\tau_E : E = (\mathfrak{s} \times TN)_{\Psi\pi} \rightarrow M = N \times \mathcal{O}_{v_0}$. Note that $E = (\mathfrak{s} \times TN)_{\Psi\pi} = (\mathfrak{s} \times TN) \times \mathcal{O}_{v_0}$ and that the anchor map $\rho_E : E = (\mathfrak{s} \times TN) \times \mathcal{O}_{v_0} \rightarrow TM = TN \times T\mathcal{O}_{v_0}$ of $\tau_E : E \rightarrow M$ is given by

$$\rho_E((\omega, \dot{u}), X_n, v) = (X_n, -\omega \cdot v) \quad (6.15)$$

for $((\omega, \dot{u}), X_n, v) \in \mathfrak{s} \times T_n N \times \mathcal{O}_{v_0}$.

Now, let $L : (\mathfrak{s} \times TN) \times \mathcal{O}_{v_0} \rightarrow \mathbb{R}$ be a Lagrangian function and D be the vector subbundle of $\tau_E : E \rightarrow M$ whose fiber $D_{(n,v)}$ over the point $(n, v) \in N \times \mathcal{O}_{v_0} = M$ is defined by

$$\begin{aligned} D_{(n,v)} &= \{((\omega, \omega \cdot v), X_n), v) / \omega \in \mathfrak{g}, X_n \in T_n N\} \\ &\subseteq E_{(n,v)} = (\mathfrak{s} \times T_n N) \times \{v\}. \end{aligned} \quad (6.16)$$

Next, we obtain the Lagrange-d'Alembert equations for the system (L, D) . For this purpose, we choose a basis $\{\omega_\alpha\}$ of \mathfrak{g} , a basis $\{u_A\}$ of V , a system of local fibred coordinates (x^i, \dot{x}^i) on TN and a system of local coordinates (v^i) on \mathcal{O}_{v_0} . Denote by ω^α (respectively, u^A) the global coordinates on \mathfrak{g} (respectively, V) induced by the basis $\{\omega_\alpha\}$ (respectively, $\{u_A\}$).

Suppose that

$$[\omega_\alpha, \omega_\beta]_{\mathfrak{g}} = c_{\alpha\beta}^\gamma \omega_\gamma, \quad \omega_\alpha \cdot u_A = a_{\alpha A}^B u_B.$$

Then, we have that

$$c_{\alpha\beta}^\gamma a_{\gamma A}^B = a_{\beta A}^C a_{\alpha C}^B - a_{\alpha A}^C a_{\beta C}^B.$$

Next, we consider the local basis of sections $\{e_i, e_\alpha, e_A\}$ of E given by

$$\begin{aligned} e_i(n, v) &= ((0, 0, \frac{\partial}{\partial x^i}|_n), v), \quad e_\alpha(n, v) = ((\omega_\alpha, \omega_\alpha \cdot v, 0_n), v) \\ e_A(n, v) &= ((0, u_A, 0_n), v) \end{aligned}$$

for $(n, v) \in N \times \mathcal{O}_{v_0} = M$. Note that $\{e_i, e_\alpha\}$ is a local basis of sections of the constraint subbundle D . In addition, if $(x^i, v^j; y^i, y^\alpha, y^A)$ are the local coordinates on E induced by the basis $\{e_i, e_\alpha, e_A\}$, it follows that

$$y^i = \dot{x}^i, \quad y^\alpha = \omega^\alpha, \quad y^A = u^A - a_{\alpha B}^A u_0^B \omega^\alpha, \quad (6.17)$$

where u_0^B is the local function on $M = N \times \mathcal{O}_{v_0}$ defined by $u_0^B = (u^B)|_{\mathcal{O}_{v_0}}$. Moreover,

$$\begin{aligned} \rho_E(e_i) &= -\frac{\partial}{\partial x^i}, \quad \rho_E(e_\alpha) = \rho_\alpha^i \frac{\partial}{\partial v^i}, \quad \rho_E(e_A) = 0, \\ [e_\alpha, e_\beta]_E &= c_{\alpha\beta}^\gamma (e_\gamma + a_{\gamma A}^B u_0^A e_B), \quad [e_\alpha, e_A]_E = -[e_A, e_\alpha]_E = a_{\alpha A}^B e_B, \end{aligned}$$

and the rest of the fundamental Lie brackets are zero. Thus, a curve

$$t \rightarrow (x^i(t), v^j(t); y^i(t), y^\alpha(t), y^A(t))$$

is a solution of the Lagrange-d'Alembert equations for the system (L, D) if and only if

$$\begin{aligned} \dot{x}^i &= y^i, \quad \dot{v}^j = \rho_\alpha^j y^\alpha, \text{ for all } i \text{ and } j, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} &= 0, \text{ for all } i, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial y^\alpha} \right) + \left(\frac{\partial L}{\partial y^\gamma} + \frac{\partial L}{\partial y^B} a_{\gamma A}^B u_0^A \right) c_{\alpha\beta}^\gamma y^\beta - \rho_\alpha^i \frac{\partial L}{\partial v^i} &= 0, \text{ for all } \alpha, \\ y^A &= 0, \text{ for all } A. \end{aligned}$$

If we consider the local expression of the curve in the coordinates $(x^i, v^j; \dot{x}^i, \omega^\alpha, u^A)$ then, from (6.17), we deduce that the above equations are equivalent to

$$\begin{aligned} \dot{x}^i &= y^i, \quad \dot{v}^j = \rho_\alpha^j \omega^\alpha, \text{ for all } i \text{ and } j, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} &= 0, \text{ for all } i, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \omega^\alpha} \right) + \frac{\partial L}{\partial \omega^\gamma} c_{\alpha\beta}^\gamma \omega^\beta + \frac{d}{dt} (a_{\alpha B}^A u_0^B \frac{\partial L}{\partial u^A}) \\ &\quad + 2a_{\gamma B}^A u_0^B \frac{\partial L}{\partial u^A} c_{\alpha\beta}^\gamma \omega^\beta - \rho_\alpha^i \frac{\partial L}{\partial v^i} = 0, \text{ for all } \alpha, \\ u^A &= a_{\alpha B}^A u_0^B \omega^\alpha, \text{ for all } A, \end{aligned}$$

or, in vector notation,

$$\begin{aligned} \dot{v} &= -\omega \cdot v, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} &= 0, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \omega} \right) + (ad_\omega^* \frac{\partial L}{\partial \omega}) &= -\frac{d}{dt} (\rho_v^* \frac{\partial L}{\partial u}) - 2ad_\omega^* (\rho_v^* \frac{\partial L}{\partial u}) - \rho_v^* \frac{\partial L}{\partial v}, \\ u &= \rho_v \omega. \end{aligned}$$

Nonholonomic Lagrangian systems, of the above type, on the left action Lie algebroid $\tau_E : E = (\mathfrak{s} \times TN) \times \mathcal{O}_{v_0} \rightarrow M = N \times \mathcal{O}_{v_0}$ may be obtained (by reduction) from an standard nonholonomic Lagrangian system with semidirect product symmetry.

In fact, let Q be the product manifold $S \times N$ and suppose that we have a Lagrangian function $\tilde{L} : TQ \rightarrow \mathbb{R}$ and a distribution \tilde{D} on Q whose characteristic space $\tilde{D}_{((g,v),n)} \subseteq T_g G \times T_v V \times T_n N \simeq T_g G \times V \times T_n N$ at the point $((g,v),n) \in S \times N$ is

$$\tilde{D}_{((g,v),n)} = \{((\dot{g}, \dot{v}), \dot{n}) \in T_g G \times V \times T_n N / \dot{v} = (T_g r_{g^{-1}})(\dot{g}) \cdot v_0\}, \quad (6.18)$$

where v_0 is a fixed point of V .

We can consider the natural left action of the Lie group S on Q and, thus, the left action A of the Lie subgroup $H_{v_0} = G_{v_0} \ltimes V$ of S on Q , where G_{v_0} is the isotropy group of v_0 with respect to the action of G on V . The tangent lift TA of A is given by

$$TA((\tilde{g}, \tilde{u}), (v_g, (v, \dot{v}), X_n)) = ((T_g l_{\tilde{g}})(v_g), (\tilde{u} + \tilde{g} \cdot v, \tilde{g} \cdot \dot{v}), X_n) \quad (6.19)$$

for $(\tilde{g}, \tilde{u}) \in H_{v_0}$ and $(v_g, (v, \dot{v}), X_n) \in T_{((g,v),n)} Q \simeq T_g G \times V \times T_n N$.

Using (6.19), it follows that the distribution \tilde{D} is invariant under the action TA of H_{v_0} on TQ . Moreover, we will assume that the Lagrangian function is also H_{v_0} -invariant. Therefore, we have a nonholonomic Lagrangian system (\tilde{L}, \tilde{D}) on the standard Lie algebroid $TQ \rightarrow Q$ which is H_{v_0} -invariant. This type of systems were considered in [64].

Since the function \tilde{L} is H_{v_0} -invariant, we deduce that there exists a real function $L : (\mathfrak{s} \times TN) \times \mathcal{O}_{v_0} \rightarrow \mathbb{R}$ on the left action Lie algebroid $\tau_E : E = (\mathfrak{s} \times TN) \times \mathcal{O}_{v_0} \rightarrow M = N \times \mathcal{O}_{v_0}$ which is defined by

$$L(((\omega, \dot{v}), X_n), u) = \tilde{L}((T_e l_g)(\omega), (v, g \cdot \dot{v}), X_n), \quad (6.20)$$

for $(((\omega, \dot{v}), X_n), u) \in \mathfrak{s} \times T_n N \times \mathcal{O}_{v_0}$, with $g \in G$, $u = g^{-1}v_0$ and $v \in V$.

Moreover, we may prove the following result.

Proposition 6.2. (1) *If $\Phi : TQ \simeq TG \times (V \times V) \times TN \rightarrow E = (\mathfrak{s} \times TN) \times \mathcal{O}_{v_0}$ and $\varphi : G \times V \times N \rightarrow N \times \mathcal{O}_{v_0}$ are the maps defined by*

$$\begin{aligned} \Phi(u_g, (v, \dot{v}), X_n) &= (((T_g l_{g^{-1}})(u_g), g^{-1} \cdot \dot{v}), X_n), g^{-1} \cdot v_0), \\ \varphi(g, v, n) &= (n, g^{-1} \cdot v_0), \end{aligned} \quad (6.21)$$

then Φ is a fiberwise bijective Lie algebroid morphism over φ .

(2) *The nonholonomic Lagrangian systems (\tilde{L}, \tilde{D}) and (L, D) on TQ and $E = (\mathfrak{s} \times TN) \times \mathcal{O}_{v_0}$ are Φ -related, that is,*

$$L \circ \Phi = \tilde{L}, \quad \Phi(\tilde{D}) = D.$$

Here, D is the vector subbundle of the vector bundle E whose fiber at the point $(n, v) \in N \times \mathcal{O}_{v_0}$ is given by (6.16).

(3) *If the system (\tilde{L}, \tilde{D}) is regular then the system (L, D) is also regular. In addition, if $\gamma : I \rightarrow TQ$ is a solution of the Lagrange-d'Alembert equations for (\tilde{L}, \tilde{D}) then $\Phi \circ \gamma : I \rightarrow (\mathfrak{s} \times TN) \times \mathcal{O}_{v_0}$ is a solution of the Lagrange-d'Alembert equations for (L, D) .*

Proof. (1) Suppose that ω_1 and ω_2 are elements of \mathfrak{g} , that \dot{v}_1 and \dot{v}_2 are vectors of V and that X_1 and X_2 are vector fields on N . Then, we consider the vector fields Z_1 and Z_2 on Q defined by

$$\begin{aligned} Z_1(g, v, n) &= (\overleftarrow{\omega}_1(g), g \cdot \dot{v}_1, X_1(n)) \in T_g G \times V \times T_n N, \\ Z_2(g, v, n) &= (\overleftarrow{\omega}_2(g), g \cdot \dot{v}_2, X_2(n)) \in T_g G \times V \times T_n N, \end{aligned}$$

for $(g, v, n) \in G \times V \times N = Q$, where $\overleftarrow{\omega}_1$ (respectively, $\overleftarrow{\omega}_2$) is the left-invariant vector field on G such that $\overleftarrow{\omega}_1(e) = \omega_1$ (respectively, $\overleftarrow{\omega}_2(e) = \omega_2$), e being the identity element of G .

A direct computation proves that

$$[Z_1, Z_2](g, v, n) = (\overleftarrow{[\omega_1, \omega_2]}_{\mathfrak{g}}(g), g(\omega_1 \cdot \dot{v}_2 - \omega_2 \cdot \dot{v}_1), [X_1, X_2](n)).$$

Moreover, if $((\omega_1, \dot{v}_1), X_1)$ (respectively, $((\omega_2, \dot{v}_2), X_2)$) is the section of the vector bundle $\tau_E : E \rightarrow M$ induced by ω_1 , \dot{v}_1 and X_1 (respectively, ω_2 , \dot{v}_2 and X_2) then it is clear that

$$\Phi \circ Z_1 = ((\omega_1, \dot{v}_1), X_1) \circ \varphi, \quad \Phi \circ Z_2 = ((\omega_2, \dot{v}_2), X_2) \circ \varphi.$$

Thus, using (6.14), it follows that

$$\Phi \circ [Z_1, Z_2] = [((\omega_1, \dot{v}_1), X_1), ((\omega_2, \dot{v}_2), X_2)]_E \circ \varphi. \quad (6.22)$$

On the other hand, we have that

$$\begin{aligned} (T_{(g, v, n)} \varphi)(u_g, \dot{v}, X_n) &= (X_n, -(T_g l_{g^{-1}})(u_g) \cdot (g^{-1} \cdot v_0)) \\ &\in T_n N \times T_{g^{-1} \cdot v_0} \mathcal{O}_{v_0} \subseteq T_n N \times V, \end{aligned}$$

for $(g, v, n) \in Q$ and $(u_g, \dot{v}, X_n) \in T_g G \times V \times T_n N \simeq T_{(g, v, n)} Q$.

Therefore, from (6.15) and (6.21), we deduce that

$$T\varphi = \rho_E \circ \Phi. \quad (6.23)$$

Consequently, using (6.22) and (6.23), we conclude that the pair (Φ, φ) is a Lie algebroid morphism. Note that one may choose a local basis $\{Z_i\}$ of vector fields on Q such that

$$Z_i(g, v, n) = (\overleftarrow{\omega}_i(g), g \cdot \dot{v}_i, X_i(n)), \quad \text{for } (g, v, n) \in Q$$

with $\omega_i \in \mathfrak{g}$, $\dot{v}_i \in V$ and $X_i \in \mathfrak{X}(N)$.

Finally, if $(g, v, n) \in Q$, it is clear that

$$\Phi|_{T_{(g,v,n)}Q} : T_{(g,v,n)}Q \simeq T_g G \times V \times T_n N \rightarrow E_{(n, g^{-1} \cdot v_0)} \simeq \mathfrak{g} \times V \times T_n N$$

is a linear isomorphism.

(2) It follows from (6.16), (6.18), (6.20) and (6.21).

(3) It follows using (1), (2) and the results of Section 4 (see Theorem 4.6). \square

The above theory may be applied to a particular example of a mechanical system: **the Chaplygin Gyro** (see [49, 64]). This system consists of a Chaplygin sphere (that is, a ball with nonhomogeneous mass distribution) with a gyro-like mechanism, consisting of a gimbal and a pendulous mass, installed in it. The gimbal is a circle-like structure such that its center coincides with the geometric center of the Chaplygin sphere. It is free to rotate about the axis connecting the north and south poles of the Chaplygin sphere. The pendulous mass can move along the smooth track of the gimbal. For this particular example, the vector space V is \mathbb{R}^3 , the Lie group G is $SO(3)$ and the manifold N is \mathbb{R}^2 . The action of $SO(3)$ on \mathbb{R}^3 is the standard one and $v_0 = (0, 0, 1)$ is the advected parameter, see [64] for more details.

6.4. Chaplygin-type systems. A frequent situation is the following. Consider a constrained Lagrangian system (L, D) on a Lie algebroid $\tau: E \rightarrow M$ such that the restriction of the anchor to the constraint distribution, $\rho|_D: D \rightarrow TM$, is an isomorphism of vector bundles. Let $h: TM \rightarrow D \subset E$ be the right-inverse of $\rho|_D$, so that $\rho \circ h = \text{id}_{TM}$. It follows that E is a transitive Lie algebroid and h is a splitting of the exact sequence

$$0 \longrightarrow \text{Ker}(\rho) \longrightarrow E \xrightarrow{\rho} TM \longrightarrow 0.$$

Let us define the function $\bar{L} \in C^\infty(TM)$ by $\bar{L} = L \circ h$. The dynamics defined by L does not reduce to the dynamics defined by \bar{L} because, while the map $\Phi = \rho$ is a morphism of Lie algebroids and $\Phi(D) = TM$, we have $\bar{L} \circ \Phi = L \circ h \circ \rho \neq L$. Nevertheless, we can use h to express the dynamics on TM , by finding relations between the dynamics defined by L and \bar{L} .

We need some auxiliary properties of the splitting h and its prolongation. We first notice that h is an admissible map over the identity in M , because $\rho_E \circ h = \text{id}_{TM}$ and $T \text{id}_M \circ \rho_{TM} = \text{id}_{TM}$, but in general h is not a morphism. We can define the tensor K , a $\ker(\rho)$ -valued differential 2-form on M , by means of

$$K(X, Y) = [h \circ X, h \circ Y] - h \circ [X, Y]$$

for every $X, Y \in \mathfrak{X}(M)$. It is easy to see that h is a morphism if and only if $K = 0$. In coordinates (x^i) in M , (x^i, v^i) in TM , and linear coordinates (x^i, y^i, y^A) on E corresponding to a local basis $\{e^i, e^A\}$ of sections of E adapted to the splitting h , we have that

$$K = \frac{1}{2} \Omega_{ij}^A dx^i \wedge dx^j \otimes e_A,$$

where Ω_{ij}^A are defined by $[e_i, e_j] = \Omega_{ij}^A e_A$.

Since h is admissible, its prolongation $\mathcal{T}^h h$ is a well-defined map from $T(TM)$ to $\mathcal{T}^E E$. Moreover, it is an admissible map, which is a morphism if and only if h is a morphism. In what respect to the energy and the Cartan 1-form, we

have that $(\mathcal{T}^h h)^* E_L = E_{\bar{L}}$ and $(\mathcal{T}^h h)^* \theta_L = \theta_{\bar{L}}$. Indeed, notice that by definition, $(\mathcal{T}^h h)^* E_L = E_L \circ h$ and

$$\begin{aligned} E_L(h(v)) &= \frac{d}{dt} L(h(v) + t(h(v)))|_{t=0} - L(h(v)) = \frac{d}{dt} L(h(v + tv))|_{t=0} - L(h(v)) \\ &= \frac{d}{dt} \bar{L}(v + tv)|_{t=0} - \bar{L}(v) = E_{\bar{L}}(v). \end{aligned}$$

On the other hand, for every $V_v \equiv (v, w, V) \in T(TM) \equiv \mathcal{T}^{TM}(TM)$ where $w = T\tau(V)$, we have

$$\begin{aligned} \langle (\mathcal{T}^h h)^* \theta_L, V \rangle &= \langle \theta_L, \mathcal{T}^h h(v, w, V) \rangle = \langle \theta_L, (h(v), h(w), Th(V)) \rangle \\ &= \frac{d}{dt} L(h(v) + t(h(w)))|_{t=0} = \frac{d}{dt} L(h(v + tw))|_{t=0} \\ &= \frac{d}{dt} \bar{L}(v + tw)|_{t=0} = \langle \theta_{\bar{L}}, V \rangle. \end{aligned}$$

Nevertheless, since h is not a morphism, and hence $(\mathcal{T}^h h)^* \circ d \neq d \circ (\mathcal{T}^h h)^*$, we have that $(\mathcal{T}^h h)^* \omega_L \neq \omega_{\bar{L}}$. Let JK be the 2-form on TM defined by

$$JK_v(V, W) = \langle J_{h(v)}, K_{h(v)}(T\tau_M(V), T\tau_M(W)) \rangle$$

where J is the momentum map defined by L and $\text{Ker } \rho$ and $V, W \in T_{h(v)}(TM)$. The notation resembles the contraction of the momentum map J with the curvature tensor K . Instead of being symplectic, the map $\mathcal{T}^h h$ satisfies

$$(\mathcal{T}^h h)^* \omega_L = \omega_{\bar{L}} + JK.$$

Indeed, we have that

$$(\mathcal{T}^h h)^* \omega_L - \omega_{\bar{L}} = [d \circ (\mathcal{T}^h h)^* - (\mathcal{T}^h h)^* \circ d] \theta_L$$

and on a pair of projectable vector fields U, V projecting onto X, Y respectively, one can easily prove that

$$[d \circ (\mathcal{T}^h h)^* - (\mathcal{T}^h h)^* \circ d] \theta_L(U, V) = \langle \theta_L, [\mathcal{T}^h h(U), \mathcal{T}^h h(V)] - \mathcal{T}^h h([U, V]) \rangle$$

from where the result follows by noticing that $\mathcal{T}^h h \circ U$ is a projectable section and projects to $h \circ X$, and similarly $\mathcal{T}^h h \circ V$ projects to $h \circ Y$. Hence $[\mathcal{T}^h h(U), \mathcal{T}^h h(V)] - \mathcal{T}^h h([U, V])$ is projectable and projects to $K(X, Y)$.

Let now Γ be the solution of the nonholonomic dynamics for (L, D) , so that Γ satisfies the equation $i_\Gamma \omega_L - dE_L \in \widehat{D}^\circ$ and the tangency condition $\Gamma|_D \in \mathcal{T}^D D$. From this second condition we deduce the existence of a vector field $\bar{\Gamma} \in \mathfrak{X}(TM)$ such that $\mathcal{T}^h h \circ \bar{\Gamma} = \Gamma \circ h$. Explicitly, the vector field $\bar{\Gamma}$ is defined by $\bar{\Gamma} = \mathcal{T}^\rho \rho \circ \Gamma \circ h$, from where it immediately follows that $\bar{\Gamma}$ is a SODE vector field on M .

Taking the pullback by $\mathcal{T}^h h$ of the first equation we get $(\mathcal{T}^h h)^*(i_\Gamma \omega_L - dE_L) = 0$ since $(\mathcal{T}^h h)^* \widehat{D}^\circ = 0$. Therefore

$$\begin{aligned} 0 &= (\mathcal{T}^h h)^* i_\Gamma \omega_L - (\mathcal{T}^h h)^* dE_L \\ &= i_{\bar{\Gamma}} (\mathcal{T}^h h)^* \omega_L - d(\mathcal{T}^h h)^* E_L \\ &= i_{\bar{\Gamma}} (\omega_{\bar{L}} + JK) - dE_{\bar{L}} \\ &= i_{\bar{\Gamma}} \omega_{\bar{L}} - dE_{\bar{L}} + i_{\bar{\Gamma}} JK. \end{aligned}$$

Therefore, the vector field $\bar{\Gamma}$ is determined by the equations

$$i_{\bar{\Gamma}} \omega_{\bar{L}} - dE_{\bar{L}} = -\langle J, K(\mathbb{T}, \cdot) \rangle,$$

where \mathbb{T} is the identity in TM considered as a vector field along the tangent bundle projection τ_M (also known as the total time derivative operator). Equivalently we can write these equations in the form

$$d_{\bar{\Gamma}} \theta_{\bar{L}} - d\bar{L} = \langle J, K(\mathbb{T}, \cdot) \rangle.$$

Note that if $\bar{a} : I \rightarrow TM$ is an integral curve of $\bar{\Gamma}$ then $a = h \circ \bar{a} : I \rightarrow D$ is a solution of the constrained dynamics for the nonholonomic Lagrangian system (L, D) on E . Conversely, if $a : I \rightarrow D$ is a solution of the constrained dynamics then $\rho \circ a : I \rightarrow TM$ is an integral curve of the vector field $\bar{\Gamma}$.

Finally we mention that extension of the above decomposition for non transitive Lie algebroids is under development.

Chaplygin systems and Atiyah algebroids. A particular case of the above theory is that of ordinary Chaplygin systems (see [4, 7, 17, 37] and references there in). In such case we have a principal G -bundle $\pi : Q \rightarrow M = Q/G$. Then, we may consider the quotient vector bundle $E = TQ/G \rightarrow M = Q/G$ and, it is well-known that, the space of sections of this vector bundle may be identified with the set of G -invariant vector fields on Q . Thus, using that the Lie bracket of two G -invariant vector fields is also G -invariant and the fact that a G -invariant vector field is π -projectable, we may define a Lie algebroid structure $([\cdot, \cdot], \rho)$ on the vector bundle $E = TQ/G \rightarrow M = Q/G$. The resultant Lie algebroid is called the **Atiyah (gauge) algebroid** associated with the principal bundle $\pi : Q \rightarrow M = Q/G$ (see [48]). Note that the canonical projection $\Phi : TQ \rightarrow E = TQ/G$ is a fiberwise bijective Lie algebroid morphism. Now, suppose that (L_Q, D_Q) is a standard nonholonomic Lagrangian system on TQ such that L_Q is G -invariant and D_Q is the horizontal distribution of a principal connection on $\pi : Q \rightarrow M = Q/G$. Then, we have a reduced nonholonomic Lagrangian system (L, D) on E . In fact, $L_Q = L \circ \Phi$ and $\Phi((D_Q)_q) = D_{\pi(q)}$, for all $q \in Q$. Moreover, $\rho|_D : D \rightarrow TM = T(Q/G)$ is an isomorphism (over the identity of M) between the vector bundles $D \rightarrow M$ and $TM \rightarrow M$. Therefore, we may apply the above general theory.

Next, we describe the nonholonomic Lagrangian system on the Atiyah algebroid associated with a particular example of a Chaplygin system: a two-wheeled planar mobile robot (see [17] and the references there in). Consider the motion of two-wheeled planar mobile robot which is able to move in the direction in which it points and, in addition, can spin about a vertical axis. Let P be the intersection point of the horizontal symmetry axis of the robot and the horizontal line connecting the centers of the two wheels. The position and orientation of the robot is determined, with respect to a fixed Cartesian reference frame by $(x, y, \theta) \in SE(2)$, where $\theta \in S^1$ is the heading angle, the coordinates $(x, y) \in \mathbb{R}^2$ locate the point P and $SE(2)$ is the group of Euclidean motions of the two-dimensional plane \mathbb{R}^2 . Let $\psi_1, \psi_2 \in S^1$ denote the rotation angles of the wheels which are assumed to be controlled independently and roll without slipping on the floor. The configuration space of the system is $Q = \mathbb{T}^2 \times SE(2)$, where \mathbb{T}^2 is the real torus of dimension 2.

The Lagrangian function L_Q is the kinetic energy corresponding to the metric g_Q

$$\begin{aligned} g_Q = & m dx \otimes dx + m dy \otimes dy + m_0 l \cos \theta (dy \otimes d\theta + d\theta \otimes dy) \\ & - m_0 l \sin \theta (dx \otimes d\theta + d\theta \otimes dx) + J d\theta \otimes d\theta + J_2 d\psi_1 \otimes d\psi_1 + J_2 d\psi_2 \otimes d\psi_2, \end{aligned}$$

where $m = m_0 + 2m_1$, m_0 is the mass of the robot without the wheels, J its momenta of inertia with respect to the vertical axis, m_1 the mass of each wheel, J_2 the axial moments of inertia of the wheels, and l the distance between the center of mass C of the robot and the point P . Thus,

$$\begin{aligned} L_Q = & \frac{1}{2} (m \dot{x}^2 + m \dot{y}^2 + 2m_0 l \dot{y} \dot{\theta} \cos \theta - 2m_0 l \dot{x} \dot{\theta} \sin \theta \\ & + J \dot{\theta}^2 + J_2 \dot{\psi}_1^2 + J_2 \dot{\psi}_2^2). \end{aligned}$$

The constraints, induced by the conditions that there is no lateral sliding of the robot and that the motion of the wheels also consists of a rolling without sliding,

are

$$\begin{aligned} \dot{x} \sin \theta - \dot{y} \cos \theta &= 0, \\ \dot{x} \cos \theta + \dot{y} \sin \theta + c\dot{\theta} + R\dot{\psi}_1 &= 0, \\ \dot{x} \cos \theta + \dot{y} \sin \theta - c\dot{\theta} + R\dot{\psi}_2 &= 0, \end{aligned}$$

where R is the radius of the wheels and $2c$ the lateral length of the robot. The constraint distribution D is then spanned by

$$\begin{aligned} \{H_1 = \frac{\partial}{\partial \psi_1} - \frac{R}{2}(\cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} + \frac{1}{c} \frac{\partial}{\partial \theta}), \\ H_2 = \frac{\partial}{\partial \psi_2} - \frac{R}{2}(\cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} - \frac{1}{c} \frac{\partial}{\partial \theta})\}, \end{aligned}$$

Note that if $\{\xi_1, \xi_2, \xi_3\}$ is the canonical basis of $\mathfrak{se}(2)$,

$$[\xi_1, \xi_2] = 0, \quad [\xi_1, \xi_3] = -\xi_2, \quad [\xi_2, \xi_3] = \xi_1,$$

then

$$H_1 = \frac{\partial}{\partial \psi_1} - \frac{R}{2} \overleftarrow{\xi}_1 - \frac{R}{2c} \overleftarrow{\xi}_3, \quad H_2 = \frac{\partial}{\partial \psi_2} - \frac{R}{2} \overleftarrow{\xi}_1 + \frac{R}{2c} \overleftarrow{\xi}_3,$$

where $\overleftarrow{\xi}_i$ ($i = 1, 2, 3$) is the left-invariant vector field of $SE(2)$ such that $\overleftarrow{\xi}_i(e) = \xi_i$, e being the identity element of $SE(2)$.

On the other hand, it is clear that $Q = \mathbb{T}^2 \times SE(2)$ is the total space of a trivial principal $SE(2)$ -bundle over $M = \mathbb{T}^2$. Moreover, the metric g_Q is $SE(2)$ -invariant and D_Q is the horizontal distribution of a principal connection on $Q = \mathbb{T}^2 \times SE(2) \rightarrow \mathbb{T}^2$.

Now, we consider the corresponding Atiyah algebroid

$$E = TQ/SE(2) \simeq (T\mathbb{T}^2 \times TSE(2))/SE(2) \rightarrow M = \mathbb{T}^2.$$

Using the left-translations on $SE(2)$, we have that the tangent bundle to $SE(2)$ may be identified with the product manifold $SE(2) \times \mathfrak{se}(2)$ and, under this identification, the Atiyah algebroid is isomorphic to the trivial vector bundle

$$\tilde{\tau}_{\mathbb{T}^2} = \tau_{\mathbb{T}^2} \circ pr_1 : T\mathbb{T}^2 \times \mathfrak{se}(2) \rightarrow \mathbb{T}^2,$$

where $\tau_{\mathbb{T}^2} : T\mathbb{T}^2 \rightarrow \mathbb{T}^2$ is the canonical projection. In addition, if $([\cdot, \cdot], \rho)$ is the Lie algebroid structure on $\tilde{\tau}_{\mathbb{T}^2} : T\mathbb{T}^2 \times \mathfrak{se}(2) \rightarrow \mathbb{T}^2$ and $\{\frac{\partial}{\partial \psi_1}, \frac{\partial}{\partial \psi_2}, \xi_1, \xi_2, \xi_3\}$ is the canonical basis of sections of $\tilde{\tau}_{\mathbb{T}^2} : T\mathbb{T}^2 \times \mathfrak{se}(2) \rightarrow \mathbb{T}^2$ then

$$\begin{aligned} \rho(\frac{\partial}{\partial \psi_1}) &= \frac{\partial}{\partial \psi_1}, & \rho(\frac{\partial}{\partial \psi_2}) &= \frac{\partial}{\partial \psi_2}, & \rho(\xi_i) &= 0, & i &= 1, 2, 3 \\ [\xi_1, \xi_3] &= -\xi_2, & [\xi_2, \xi_3] &= \xi_1, \end{aligned}$$

and the rest of the fundamental Lie brackets are zero.

Denote by $(\psi_1, \psi_2, \dot{\psi}_1, \dot{\psi}_2, \omega^1, \omega^2, \omega^3)$ the (local) coordinates on $T\mathbb{T}^2 \times \mathfrak{se}(2)$ induced by the basis $\{\frac{\partial}{\partial \psi_1}, \frac{\partial}{\partial \psi_2}, \xi_1, \xi_2, \xi_3\}$. Then, the reduced Lagrangian $L : T\mathbb{T}^2 \times \mathfrak{se}(2) \rightarrow \mathbb{R}$ is given by

$$L = \frac{1}{2}(m(\omega^1)^2 + m(\omega^2)^2 + 2m_0 l \omega^2 \omega^3 + J(\omega^3)^2 + J_2 \dot{\psi}_1^2 + J_2 \dot{\psi}_2^2)$$

and the constraint vector subbundle D is generated by the sections

$$e_1 = \frac{\partial}{\partial \psi_1} - \frac{R}{2} \xi_1 - \frac{R}{2c} \xi_3, \quad e_2 = \frac{\partial}{\partial \psi_2} - \frac{R}{2} \xi_1 + \frac{R}{2c} \xi_3.$$

Since the system (L_Q, D_Q) is regular on the standard Lie algebroid $\tau_Q : TQ \rightarrow Q$, we deduce that the nonholonomic Lagrangian system (L, D) on the Atiyah algebroid $\tilde{\tau}_{\mathbb{T}^2} : T\mathbb{T}^2 \times \mathfrak{se}(2) \rightarrow \mathbb{T}^2$ is also regular.

Now, as in Section 3, we consider a basis of sections of $\tilde{\tau}_{\mathbb{T}^2} : T\mathbb{T}^2 \times \mathfrak{se}(2) \rightarrow \mathbb{T}^2$ which is adapted to the constraint subbundle D . This basis is

$$\{e_1, e_2, \xi_1, \xi_2, \xi_3\}.$$

The corresponding (local) coordinates on $T\mathbb{T}^2 \times \mathfrak{se}(2)$ are $(\psi_1, \psi_2, \dot{\psi}_1, \dot{\psi}_2, \tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3)$, where

$$\omega^1 = \tilde{\omega}^1 - \frac{R}{2}\dot{\psi}_1 - \frac{R}{2}\dot{\psi}_2, \quad \omega^2 = \tilde{\omega}^2, \quad \omega^3 = \tilde{\omega}^3 - \frac{R}{2c}\dot{\psi}_1 + \frac{R}{2c}\dot{\psi}_2.$$

Therefore, using (3.7), we deduce that the Lagrange-d'Alembert equations for the system (L, D) are

$$\begin{aligned} \ddot{\psi}_1 &= \frac{U(\dot{\psi}_2 - \dot{\psi}_1)}{P^2 - S^2}(P\dot{\psi}_2 + S\dot{\psi}_1), & \ddot{\psi}_2 &= -\frac{U(\dot{\psi}_2 - \dot{\psi}_1)}{P^2 - S^2}(P\dot{\psi}_1 + S\dot{\psi}_2), \\ \tilde{\omega}^1 &= \tilde{\omega}^2 = \tilde{\omega}^3 = 0, \end{aligned}$$

where P, S and U are the real numbers

$$P = \frac{R^2}{4}(m + \frac{J}{c^2}) + J_2, \quad S = \frac{R^2}{4}(m - \frac{J}{c^2}), \quad U = \frac{R^3}{4c^2}m_0l.$$

On the other hand, the Lagrangian function $\bar{L} : T\mathbb{T}^2 \rightarrow \mathbb{R}$ on $T\mathbb{T}^2$ is given by

$$\bar{L}(\psi_1, \psi_2, \dot{\psi}_1, \dot{\psi}_2) = \frac{1}{2}(P\dot{\psi}_1^2 + P\dot{\psi}_2^2 + 2S\dot{\psi}_1\dot{\psi}_2)$$

and the 1-form $\langle J, K(\mathbb{T}, \cdot) \rangle$ on $T\mathbb{T}^2$ is

$$\langle J, K(\mathbb{T}, \cdot) \rangle = -U(\dot{\psi}_2 - \dot{\psi}_1)(\dot{\psi}_1 d\psi_2 - \dot{\psi}_2 d\psi_1).$$

7. NONLINEARLY CONSTRAINED LAGRANGIAN SYSTEMS

We show in this section how the main results for linearly constrained Lagrangian systems can be extended to the case of Lagrangian systems with nonlinear nonholonomic constraints. This is true under the assumption that a suitable version of the classical Chetaev's principle in nonholonomic mechanics is valid (see e.g., [41] for the study of standard nonholonomic Lagrangian systems subject to nonlinear constraints).

Let $\tau : E \rightarrow M$ be a Lie algebroid and \mathcal{M} be a submanifold of E such that $\pi = \tau|_{\mathcal{M}} : \mathcal{M} \rightarrow M$ is a fibration. \mathcal{M} is the constraint submanifold. Since π is a fibration, the prolongation $\mathcal{T}^E\mathcal{M}$ is well-defined. We will denote by r the dimension of the fibers of $\pi : \mathcal{M} \rightarrow M$, that is, $r = \dim \mathcal{M} - \dim M$.

We define the bundle $\mathcal{V} \rightarrow \mathcal{M}$ of **virtual displacements** as the subbundle of τ^*E of rank r whose fiber at a point $a \in \mathcal{M}$ is

$$\mathcal{V}_a = \{b \in E_{\tau(a)} \mid b_a^V \in T_a\mathcal{M}\}.$$

In other words, the elements of \mathcal{V} are pairs of elements $(a, b) \in E \oplus E$ such that

$$\left. \frac{d}{dt}\phi(a + tb) \right|_{t=0} = 0,$$

for every local constraint function ϕ .

We also define the bundle of **constraint forces** Ψ by $\Psi = S^*((\mathcal{T}^E\mathcal{M})^\circ)$, in terms of which we set the Lagrange-d'Alembert equations for a regular Lagrangian function $L \in C^\infty(E)$ as follows:

$$\begin{aligned} (i_\Gamma \omega_L - dE_L)|_{\mathcal{M}} &\in \text{Sec}(\Psi), \\ \Gamma|_{\mathcal{M}} &\in \text{Sec}(\mathcal{T}^E\mathcal{M}), \end{aligned} \tag{7.1}$$

the unknown being the section Γ . The above equations reproduce the corresponding ones for standard nonlinear constrained systems.

From (2.8) and (7.1), it follows that

$$(i_{S\Gamma}\omega_L - i_{\Delta}\omega_L)|_{\mathcal{M}} = 0,$$

which implies that a solution Γ of equations (7.1) is a SODE section along \mathcal{M} , that is, $(S\Gamma - \Delta)|_{\mathcal{M}} = 0$.

Note that the rank of the vector bundle $(\mathcal{T}^E\mathcal{M})^\circ \rightarrow \mathcal{M}$ is $s = \text{rank}(E) - r$ and, since π is a fibration, the transformation $S^* : (\mathcal{T}^E\mathcal{M})^\circ \rightarrow \Psi$ defines an isomorphism between the vector bundles $(\mathcal{T}^E\mathcal{M})^\circ \rightarrow \mathcal{M}$ and $\Psi \rightarrow \mathcal{M}$. Therefore, the rank of Ψ is also s . Moreover, if $a \in \mathcal{M}$ we have

$$\Psi_a = S^*((\mathcal{T}_a^E\mathcal{M})^\circ) = \{\zeta \circ \mathcal{T}\tau \mid \zeta \in \mathcal{V}_a^\circ\}. \quad (7.2)$$

In fact, if $\alpha_a \in (\mathcal{T}_a^E\mathcal{M})^\circ$, we may define $\zeta \in E_{\tau(a)}^*$ by

$$\zeta(b) = \alpha_a(\xi^V(a, b)), \text{ for } b \in E_{\tau(a)}.$$

Then, a direct computation proves that $\zeta \in \mathcal{V}_a^\circ$ and $S^*(\alpha_a) = \zeta \circ \mathcal{T}\tau$. Thus, we obtain

$$\Psi_a \subseteq \{\zeta \circ \mathcal{T}\tau \mid \zeta \in \mathcal{V}_a^\circ\}$$

and, using that the dimension of both spaces is s , we deduce (7.2) holds. Note that, in the particular case when the constraints are linear, we have $\mathcal{V} = \tau^*(D)$ and $\Psi = \widetilde{D}^\circ$.

Next, we consider the vector bundles F and $\mathcal{T}^\mathcal{V}\mathcal{M}$ over \mathcal{M} whose fibers at the point $a \in \mathcal{M}$ are

$$F_a = \omega_L^{-1}(\Psi_a), \quad \mathcal{T}_a^\mathcal{V}\mathcal{M} = \{(b, v) \in \mathcal{V}_a \times T_a\mathcal{M} \mid T\pi(v) = \rho(b)\}.$$

It follows that

$$F_a = \{z \in \mathcal{T}_a^E E \mid \text{exists } \zeta \in \mathcal{V}_a^\circ \text{ and } i_z\omega_L(a) = \zeta \circ \mathcal{T}\tau\}$$

and

$$\mathcal{T}_a^\mathcal{V}\mathcal{M} = \{z \in \mathcal{T}_a^E \mathcal{M} \mid T\pi(z) \in \mathcal{V}_a\} = \{z \in \mathcal{T}_a^E \mathcal{M} \mid S(z) \in \mathcal{T}_a^E \mathcal{M}\}. \quad (7.3)$$

Note that the dimension of $\mathcal{T}_a^\mathcal{V}\mathcal{M}$ is $2r$ and, when the constraints are linear, i.e., \mathcal{M} is a vector subbundle D of E , we obtain

$$\mathcal{T}_a^\mathcal{V}\mathcal{M} = \mathcal{T}_a^D D, \text{ for all } a \in \mathcal{M} = D.$$

Moreover, from (7.3), we deduce that the vertical lift of an element of \mathcal{V} is an element of $\mathcal{T}^\mathcal{V}\mathcal{M}$. Thus we can define for $b, c \in \mathcal{V}_a$

$$G_a^{L, \mathcal{V}}(b, c) = \omega_L(a)(\tilde{b}, \xi^V(a, c)),$$

where $\tilde{b} \in \mathcal{T}_a^E E$ and $\mathcal{T}\tau(\tilde{b}) = b$.

7.1. Dynamics in local coordinates. Here we analyze the local nature of equations (7.1). We consider local coordinates (x^i) on an open subset U of M and take a basis $\{e_\alpha\}$ of local sections of E . In this way, we have local coordinates (x^i, y^α) on E . Suppose that the local equations defining \mathcal{M} as a submanifold of E are

$$\phi^A = 0, \quad A = 1, \dots, s,$$

where ϕ^A are independent local constraint functions. Since $\pi : \mathcal{M} \rightarrow M$ is a fibration, it follows that the matrix $(\frac{\partial \phi^A}{\partial y^\alpha})$ is of rank s . Thus, if d is the differential of the Lie algebroid $\mathcal{T}^E E \rightarrow E$, we deduce that $\{d\phi^A|_{\mathcal{M}}\}_{A=1, \dots, s}$ is a local basis of sections of the vector bundle $(\mathcal{T}^E \mathcal{M})^0 \rightarrow \mathcal{M}$. Note that

$$d\phi^A = \rho_\alpha^j \frac{\partial \phi^A}{\partial x^j} \mathcal{X}^\alpha + \frac{\partial \phi^A}{\partial y^\alpha} \mathcal{V}^\alpha.$$

Moreover, $\{S^*(d\phi^A)|_{\mathcal{M}} = \frac{\partial\phi^A}{\partial y^\alpha}\mathcal{X}^\alpha|_{\mathcal{M}}\}_{A=1,\dots,s}$ is a local basis of sections of the vector bundle $\Psi \rightarrow \mathcal{M}$.

Next, we introduce the local sections $\{Z_A\}_{A=1,\dots,s}$ of $\mathcal{T}^E E \rightarrow E$ defined by

$$i_{Z_A}\omega_L = S^*(d\phi^A) = \frac{\partial\phi^A}{\partial y^\alpha}\mathcal{X}^\alpha.$$

A direct computation, using (2.5), proves that

$$Z_A = -\frac{\partial\phi^A}{\partial y^\alpha}W^{\alpha\beta}\mathcal{V}_\beta, \quad \text{for all } A, \quad (7.4)$$

where $(W^{\alpha\beta})$ is the inverse matrix of $(W_{\alpha\beta} = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta})$. Furthermore, it is clear that $\{Z_A|_{\mathcal{M}}\}$ is a local basis of sections of the vector bundle $F \rightarrow \mathcal{M}$.

On the other hand, if Γ_L is the Euler-Lagrange section associated with the regular Lagrangian L , then a section Γ of $\mathcal{T}^E \mathcal{M} \rightarrow \mathcal{M}$ is a solution of equations (7.1) if and only if

$$\Gamma = (\Gamma_L + \lambda^A Z_A)|_{\mathcal{M}}$$

with λ^A local real functions on E satisfying

$$(\lambda^A d\phi^B(Z_A) + d\phi^B(\Gamma_L))|_{\mathcal{M}} = 0, \quad \text{for all } B = 1, \dots, s.$$

Therefore, using (7.4), we conclude that there exists a unique solution of the Lagrange-d'Alembert equations (7.1) if and only if the matrix

$$(\mathcal{C}^{AB} = \frac{\partial\phi^A}{\partial y^\alpha}W^{\alpha\beta}\frac{\partial\phi^B}{\partial y^\beta})_{A,B=1,\dots,s} \quad (7.5)$$

is regular. We are now ready to prove the following result.

Theorem 7.1. *The following properties are equivalent:*

- (1) *The constrained Lagrangian system (L, \mathcal{M}) is regular, that is, there exists a unique solution of the Lagrange-d'Alembert equations,*
- (2) $\text{Ker } G^{L,\nu} = \{0\},$
- (3) $\mathcal{T}^E \mathcal{M} \cap F = \{0\},$
- (4) $\mathcal{T}^\nu \mathcal{M} \cap (\mathcal{T}^\nu \mathcal{M})^\perp = \{0\}.$

Proof. It is clear that the matrix (\mathcal{C}^{AB}) in (7.5) is regular if and only if $\mathcal{T}^E \mathcal{M} \cap F = \{0\}$. Thus, the properties (1) and (3) are equivalent. Moreover, proceeding as in the proof of Theorem 3.12, we deduce that the properties (2) and (3) (respectively, (2) and (4)) also are equivalent. \square

Remark 7.2 (Lagrangians of mechanical type). If L is a Lagrangian function of mechanical type, then, using Theorem 7.1, we deduce (as in the case of linear constraints) that the constrained system (L, \mathcal{M}) is always regular. \bullet

7.2. Lagrange-d'Alembert solutions and nonholonomic bracket. Assume that the constrained Lagrangian system (L, \mathcal{M}) is regular. Then (3) in Theorem 7.1 is equivalent to $(\mathcal{T}^E E)|_{\mathcal{M}} = \mathcal{T}^E \mathcal{M} \oplus F$. Denote by P and Q the complementary projectors defined by this decomposition

$$P_a : \mathcal{T}_a^E E \rightarrow \mathcal{T}_a^E \mathcal{M}, \quad Q_a : \mathcal{T}_a^E E \rightarrow F_a, \quad \text{for all } a \in \mathcal{M}.$$

As in the case of linear constraints, we may prove the following.

Theorem 7.3. *Let (L, \mathcal{M}) be a regular constrained Lagrangian system and let Γ_L be the solution of the free dynamics, i.e., $i_{\Gamma_L}\omega_L = dE_L$. Then, the solution of the constrained dynamics is the SODE $\Gamma_{(L, \mathcal{M})}$ obtained as follows*

$$\Gamma_{(L, \mathcal{M})} = P(\Gamma_L|_{\mathcal{M}}).$$

On the other hand, (4) in Theorem 7.1 is equivalent to $(\mathcal{T}^E E)|_{\mathcal{M}} = \mathcal{T}^{\mathcal{V}}\mathcal{M} \oplus (\mathcal{T}^{\mathcal{V}}\mathcal{M})^{\perp}$ and we will denote by \bar{P} and \bar{Q} the corresponding projectors induced by this decomposition, that is,

$$\bar{P}_a : \mathcal{T}_a^E E \rightarrow \mathcal{T}_a^{\mathcal{V}}\mathcal{M}, \quad \bar{Q}_a : \mathcal{T}_a^E E \rightarrow (\mathcal{T}_a^{\mathcal{V}}\mathcal{M})^{\perp}, \text{ for all } a \in \mathcal{M}.$$

Theorem 7.4. *Let (L, \mathcal{M}) be a regular constrained Lagrangian system, Γ_L (respectively, $\Gamma_{(L, \mathcal{M})}$) be the solution of the free (respectively, constrained) dynamics and Δ be the Liouville section of $\mathcal{T}^E E \rightarrow E$. Then, $\Gamma_{(L, \mathcal{M})} = \bar{P}(\Gamma_L|_{\mathcal{M}})$ if and only if the restriction to \mathcal{M} of the vector field $\rho^1(\Delta)$ on E is tangent to \mathcal{M} .*

Proof. Proceeding as in the proof of Lemma 3.11, we obtain that

$$(\mathcal{T}_a^{\mathcal{V}}\mathcal{M})^{\perp} \cap \text{Ver}(\mathcal{T}_a^E E) = F_a, \text{ for all } a \in \mathcal{M}.$$

Thus, it is clear that

$$Q(\Gamma_L(a)) \in F_a \subseteq (\mathcal{T}_a^{\mathcal{V}}\mathcal{M})^{\perp}, \text{ for all } a \in \mathcal{M}.$$

Moreover, from (7.3) and using the fact that the solution of the constrained dynamics is a SODE along \mathcal{M} , we deduce

$$\Gamma_{(L, \mathcal{M})}(a) = P(\Gamma_L(a)) \in \mathcal{T}_a^{\mathcal{V}}\mathcal{M}, \text{ for all } a \in \mathcal{M},$$

if and only if the restriction to \mathcal{M} of the vector field $\rho^1(\Delta)$ on E is tangent to \mathcal{M} . This proves the result. \square

Remark 7.5 (Linear constraints). Note that if \mathcal{M} is a vector subbundle D of E , then the vector field $\rho^1(\Delta)$ is always tangent to $\mathcal{M} = D$. \bullet

As in the case of linear constraints, one may develop the distributional approach in order to obtain the solution of the constrained dynamics. In fact, if (L, \mathcal{M}) is regular, then $\mathcal{T}^{\mathcal{V}}\mathcal{M} \rightarrow \mathcal{M}$ is a symplectic subbundle of $(\mathcal{T}^E E, \omega_L)$ and, thus, the restriction $\omega^{L, \mathcal{M}}$ of ω_L to $\mathcal{T}^{\mathcal{V}}\mathcal{M}$ is a symplectic section on that bundle. We may also define $\varepsilon^{L, \mathcal{M}}$ as the restriction of dE_L to $\mathcal{T}^{\mathcal{V}}\mathcal{M}$. Then, taking the restriction of Lagrange-d'Alembert equations to $\mathcal{T}^{\mathcal{V}}\mathcal{M}$, we get the following equation

$$i_{\bar{\Gamma}}\omega^{L, \mathcal{M}} = \varepsilon^{L, \mathcal{M}}, \quad (7.6)$$

which uniquely determines a section $\bar{\Gamma}$ of $\mathcal{T}^{\mathcal{V}}\mathcal{M} \rightarrow \mathcal{M}$. It is not difficult to prove that $\bar{\Gamma} = \bar{P}(\Gamma_L|_{\mathcal{M}})$. Thus, the unique solution of equation (7.6) is the solution of the constrained dynamics if and only if the vector field $\rho^1(\Delta)$ is tangent to \mathcal{M} .

Let (L, \mathcal{M}) a regular constrained Lagrangian system. Since $S^* : (\mathcal{T}^E \mathcal{M})^0 \rightarrow \Psi$ is a vector bundle isomorphism, it follows that there exists a unique section $\alpha_{(L, \mathcal{M})}$ of $(\mathcal{T}^E \mathcal{M})^0 \rightarrow \mathcal{M}$ such that

$$i_{Q(\Gamma_L|_{\mathcal{M}})}\omega_L = S^*(\alpha_{(L, \mathcal{M})}).$$

Moreover, we have the following result.

Theorem 7.6. *If (L, \mathcal{M}) is a regular constrained Lagrangian system and $\Gamma_{(L, \mathcal{M})}$ is the solution of the dynamics, then $d_{\Gamma_{(L, \mathcal{M})}}(E_L|_{\mathcal{M}}) = 0$ if and only if $\alpha_{(L, \mathcal{M})}(\Delta|_{\mathcal{M}}) = 0$. In particular, if the vector field $\rho^1(\Delta)$ is tangent to \mathcal{M} , then $d_{\Gamma_{(L, \mathcal{M})}}(E_L|_{\mathcal{M}}) = 0$.*

Proof. From Theorem 7.3, we deduce

$$(i_{\Gamma_{(L, \mathcal{M})}}\omega_L - dE_L)|_{\mathcal{M}} = -S^*(\alpha_{(L, \mathcal{M})}).$$

Therefore, using that $\Gamma_{(L, \mathcal{M})}$ is a SODE along \mathcal{M} , we obtain

$$d_{\Gamma_{(L, \mathcal{M})}}(E_L|_{\mathcal{M}}) = \alpha_{(L, \mathcal{M})}(\Delta|_{\mathcal{M}}).$$

\square

Now, let (L, \mathcal{M}) be a regular constrained Lagrangian system. In addition, suppose that f and g are two smooth functions on \mathcal{M} and take arbitrary extensions to E denoted by the same letters. Then, as in Section 3.5, we may define *the nonholonomic bracket* of f and g as follows

$$\{f, g\}_{nh} = \omega_L(\bar{P}(X_f), \bar{P}(X_g))|_{\mathcal{M}},$$

where X_f and X_g are the Hamiltonian sections on $\mathcal{T}^E E$ associated with f and g , respectively.

Moreover, proceeding as in the case of linear constraints, one can prove that

$$\dot{f} = \rho^1(R_L)(f) + \{f, E_L\}_{nh}, \quad f \in C^\infty(\mathcal{M}),$$

where R_L is the section of $\mathcal{T}^E \mathcal{M} \rightarrow \mathcal{M}$ defined by $R_L = P(\Gamma_L|_{\mathcal{M}}) - \bar{P}(\Gamma_L|_{\mathcal{M}})$. Thus, in the particular case when the restriction to \mathcal{M} of the vector field $\rho^1(\Delta)$ on E is tangent to \mathcal{M} , it follows that

$$\dot{f} = \{f, E_L\}_{nh}, \quad \text{for } f \in C^\infty(\mathcal{M}).$$

Alternatively, since $\mathcal{T}^\nu \mathcal{M}$ is an anchored vector bundle, we may consider the differential $\bar{d}f \in \text{Sec}((\mathcal{T}^\nu \mathcal{M})^*)$ for a function $f \in C^\infty(\mathcal{M})$. Thus, since the restriction $\omega^{L, \mathcal{M}}$ of ω_L to $\mathcal{T}^\nu \mathcal{M}$ is regular, we have a unique section $\bar{X}_f \in \text{Sec}(\mathcal{T}^\nu \mathcal{M})$ given by $i_{\bar{X}_f} \omega^{L, \mathcal{M}} = \bar{d}f$ and it follows that

$$\{f, g\}_{nh} = \omega^{L, \mathcal{M}}(\bar{X}_f, \bar{X}_g).$$

7.3. Morphisms and reduction. Let (L, \mathcal{M}) be a regular constrained Lagrangian system on a Lie algebroid $\tau : E \rightarrow M$ and let (L', \mathcal{M}') be another constrained Lagrangian system on a second Lie algebroid $\tau' : E' \rightarrow M'$. Suppose also that we have a fiberwise surjective morphism of Lie algebroids $\Phi : E \rightarrow E'$ over a surjective submersion $\phi : M \rightarrow M'$ such that:

- (i) $L = L' \circ \Phi$,
- (ii) $\Phi|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}'$ is a surjective submersion,
- (iii) $\Phi(\mathcal{V}_a) = \mathcal{V}'_{\Phi(a)}$, for all $a \in \mathcal{M}$.

Note that condition (ii) implies that $\Phi(\mathcal{V}_a) \subseteq \mathcal{V}'_{\Phi(a)}$, for all $a \in \mathcal{M}$. Moreover, if $V(\Phi)$ is the vertical bundle of Φ and

$$V_a(\Phi) \subset T_a \mathcal{M}, \quad \text{for all } a \in \mathcal{M},$$

then condition (ii) also implies that $\mathcal{V}'_{\Phi(a)} \subseteq \Phi(\mathcal{V}_a)$, for all $a \in \mathcal{M}$.

On the other hand, using condition (iii) and Proposition 4.2, it follows that $\ker G^{L', \mathcal{V}'} = \{0\}$ and, thus, the constrained Lagrangian system (L', \mathcal{M}') is regular. Moreover, proceeding as in the proof of Lemma 4.5 and Theorem 4.6, we deduce the following results.

Lemma 7.7. *With respect to the decompositions*

$$(\mathcal{T}^E E)|_{\mathcal{M}} = \mathcal{T}^E \mathcal{M} \oplus F \quad \text{and} \quad (\mathcal{T}^{E'} E')|_{\mathcal{M}'} = \mathcal{T}^{E'} \mathcal{M}' \oplus F'$$

we have the following properties

- (1) $\mathcal{T}^\Phi \Phi(\mathcal{T}^E \mathcal{M}) = \mathcal{T}^{E'} \mathcal{M}'$,
- (2) $\mathcal{T}^\Phi \Phi(F) = F'$,
- (3) *If P, Q and P', Q' are the projectors associated with (L, \mathcal{M}) and (L', \mathcal{M}') , respectively, then $P' \circ \mathcal{T}^\Phi \Phi = \mathcal{T}^\Phi \Phi \circ P$ and $Q' \circ \mathcal{T}^\Phi \Phi = \mathcal{T}^\Phi \Phi \circ Q$.*

With respect to the decompositions

$$(\mathcal{T}^E E)|_{\mathcal{M}} = \mathcal{T}^\nu \mathcal{M} \oplus (\mathcal{T}^\nu \mathcal{M})^\perp \quad \text{and} \quad (\mathcal{T}^{E'} E')|_{\mathcal{M}'} = \mathcal{T}^{\nu'} \mathcal{M}' \oplus (\mathcal{T}^{\nu'} \mathcal{M}')^\perp$$

we have the following properties

- (4) $(\mathcal{T}^\Phi \Phi)(\mathcal{T}^\nu \mathcal{M}) = \mathcal{T}^{\nu'} \mathcal{M}'$,

- (5) $(\mathcal{T}^\Phi \Phi)((\mathcal{T}^\vee \mathcal{M})^\perp) = (\mathcal{T}^{\vee'} \mathcal{M}')^\perp$,
 (6) If \bar{P}, \bar{Q} and \bar{P}', \bar{Q}' are the projectors associated with (L, \mathcal{M}) and (L', \mathcal{M}') , respectively, then $\bar{P}' \circ \mathcal{T}^\Phi \Phi = \mathcal{T}^\Phi \Phi \circ \bar{P}$ and $\bar{Q}' \circ \mathcal{T}^\Phi \Phi = \mathcal{T}^\Phi \Phi \circ \bar{Q}$.

Theorem 7.8 (Reduction of the constrained dynamics). *Let (L, \mathcal{M}) be a regular constrained Lagrangian system on a Lie algebroid E and let (L', \mathcal{M}') be a constrained Lagrangian system on a second Lie algebroid E' . Assume that we have a fiberwise surjective morphism of Lie algebroids $\Phi : E \rightarrow E'$ over $\phi : M \rightarrow M'$ such that conditions (i)-(iii) hold. If $\Gamma_{(L, \mathcal{M})}$ is the constrained dynamics for L and $\Gamma_{(L', \mathcal{M}')}$ is the constrained dynamics for L' , respectively, then $\mathcal{T}^\Phi \Phi \circ \Gamma_{(L, \mathcal{M})} = \Gamma_{(L', \mathcal{M}')} \circ \Phi$. If $a(t)$ is a solution of Lagrange-d'Alembert differential equations for L , then $\Phi(a(t))$ is a solution of Lagrange-d'Alembert differential equations for L' .*

We will say that the constrained dynamics $\Gamma_{(L', \mathcal{M}')}$ is **the reduction of the constrained dynamics** $\Gamma_{(L, \mathcal{M})}$ by the morphism Φ . As in the case of linear constraints (see Theorem 4.7), we also may prove the following result

Theorem 7.9. *Under the same hypotheses as in Theorem 7.8, we have that*

$$\{f' \circ \Phi, g' \circ \Phi\}_{nh} = \{f', g'\}'_{nh} \circ \Phi,$$

for $f', g' \in C^\infty(\mathcal{M}')$, where $\{\cdot, \cdot\}_{nh}$ (respectively, $\{\cdot, \cdot\}'_{nh}$) is the nonholonomic bracket for the constrained system (L, \mathcal{M}) (respectively, (L', \mathcal{M}')). In other words, $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ is an almost-Poisson morphism.

Now, let $\phi : Q \rightarrow M$ be a principal G -bundle and $\tau : E \rightarrow Q$ be a Lie algebroid over Q . In addition, assume that we have an action of G on E such that the quotient vector bundle E/G is defined and the set $\text{Sec}(E)^G$ of equivariant sections of E is a Lie subalgebra of $\text{Sec}(E)$. Then, $E' = E/G$ has a canonical Lie algebroid structure over M such that the canonical projection $\Phi : E \rightarrow E'$ is a fiberwise bijective Lie algebroid morphism over ϕ (see Theorem 4.8).

Next, suppose that (L, \mathcal{M}) is a G -invariant regular constrained Lagrangian system, that is, the Lagrangian function L and the constraint submanifold \mathcal{M} are G -invariant. Then, one may define a Lagrangian function $L' : E' \rightarrow \mathbb{R}$ on E' such that

$$L = L' \circ \Phi.$$

Moreover, G acts on \mathcal{M} and if the set of orbits $\mathcal{M}' = \mathcal{M}/G$ of this action is a quotient manifold, that is, \mathcal{M}' is a smooth manifold and the canonical projection $\Phi|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}' = \mathcal{M}/G$ is a submersion, then one may consider the constrained Lagrangian system (L', \mathcal{M}') on E' .

Remark 7.10 (Quotient manifold). If \mathcal{M} is a closed submanifold of E , then, using a well-known result (see [1, Theorem 4.1.20]), it follows that the set of orbits $\mathcal{M}' = \mathcal{M}/G$ is a quotient manifold. •

Since the orbits of the action of G on E are the fibers of Φ and \mathcal{M} is G -invariant, we deduce that

$$V_a(\Phi) \subseteq T_a \mathcal{M}, \text{ for all } a \in \mathcal{M},$$

which implies that $\Phi|_{\mathcal{V}_a} : \mathcal{V}_a \rightarrow \mathcal{V}'_{\Phi(a)}$ is a linear isomorphism, for all $a \in \mathcal{M}$.

Thus, from Theorem 7.8, we conclude that the constrained Lagrangian system (L', \mathcal{M}') is regular and that

$$\mathcal{T}^\Phi \Phi \circ \Gamma_{(L, \mathcal{M})} = \Gamma_{(L', \mathcal{M}')} \circ \Phi,$$

where $\Gamma_{(L, \mathcal{M})}$ (resp., $\Gamma_{(L', \mathcal{M}')}$) is the constrained dynamics for L (resp., L'). In addition, using Theorem 7.9, we obtain that $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ is an almost-Poisson morphism when on \mathcal{M} and \mathcal{M}' we consider the almost-Poisson structures induced by the corresponding nonholonomic brackets.

We illustrate the results above in a particular example in the following subsection.

7.4. Example: a ball rolling on a rotating table. The following example is taken from [4, 11, 57]. A (homogeneous) sphere of radius $r > 0$, unit mass $m = 1$ and inertia about any axis k^2 , rolls without sliding on a horizontal table which rotates with constant angular velocity Ω about a vertical axis through one of its points. Apart from the constant gravitational force, no other external forces are assumed to act on the sphere.

Choose a Cartesian reference frame with origin at the center of rotation of the table and z -axis along the rotation axis. Let (x, y) denote the position of the point of contact of the sphere with the table. The configuration space for the sphere on the table is $Q = \mathbb{R}^2 \times SO(3)$, where $SO(3)$ may be parameterized by the Eulerian angles θ, φ and ψ . The kinetic energy of the sphere is then given by

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + k^2(\dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta)),$$

and with the potential energy being constant, we may put $V = 0$. The constraint equations are

$$\begin{aligned} \dot{x} - r\dot{\theta}\sin\psi + r\dot{\varphi}\sin\theta\cos\psi &= -\Omega y, \\ \dot{y} + r\dot{\theta}\cos\psi + r\dot{\varphi}\sin\theta\sin\psi &= \Omega x. \end{aligned}$$

Since the Lagrangian function is of mechanical type, the constrained system is regular. Note that the constraints are affine, and hence not linear, and that the restriction to the constraint submanifold \mathcal{M} of the Liouville vector field on TQ is not tangent to \mathcal{M} . Indeed, the constraints are linear if and only if $\Omega = 0$.

Now, we can proceed from here to construct the equations of motion of the sphere, following the general theory. However, the use of the Eulerian angles as part of the coordinates leads to very complicated expressions. Instead, one may choose to exploit the symmetry of the problem, and one way to do this is by the use of appropriate *quasi-coordinates* (see [57]). First of all, observe that the kinetic energy may be expressed as

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)),$$

where

$$\begin{aligned} \omega_x &= \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi, \\ \omega_y &= \dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi, \\ \omega_z &= \dot{\varphi}\cos\theta + \dot{\psi}, \end{aligned}$$

are the components of the angular velocity of the sphere. The constraint equations expressing the rolling conditions can be rewritten as

$$\begin{aligned} \dot{x} - r\omega_y &= -\Omega y, \\ \dot{y} + r\omega_x &= \Omega x. \end{aligned}$$

Next, following [11], we consider local coordinates $(\bar{x}, \bar{y}, \bar{\theta}, \bar{\varphi}, \bar{\psi}; \pi_i)_{i=1, \dots, 5}$ on $TQ = T\mathbb{R}^2 \times T(SO(3))$, where

$$\begin{aligned} \bar{x} &= x, \quad \bar{y} = y, \quad \bar{\theta} = \theta, \quad \bar{\varphi} = \varphi, \quad \bar{\psi} = \psi, \\ \pi_1 &= r\dot{x} + k^2\dot{q}_2, \quad \pi_2 = r\dot{y} - k^2\dot{q}_1, \quad \pi_3 = k^2\dot{q}_3, \\ \pi_4 &= \frac{k^2}{(k^2 + r^2)}(\dot{x} - r\dot{q}_2 + \Omega y), \quad \pi_5 = \frac{k^2}{(k^2 + r^2)}(\dot{y} + r\dot{q}_1 - \Omega x), \end{aligned}$$

and $(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ are the quasi-coordinates defined by

$$\dot{q}_1 = \omega_x, \quad \dot{q}_2 = \omega_y, \quad \dot{q}_3 = \omega_z.$$

As is well-known, the coordinates q_i only have a symbolic meaning. In fact, $\{\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial q_3}\}$ is the basis of left-invariant vector fields on $SO(3)$ given by

$$\begin{aligned}\frac{\partial}{\partial q_1} &= (\cos \psi) \frac{\partial}{\partial \theta} + \frac{\sin \psi}{\sin \theta} \left(\frac{\partial}{\partial \varphi} - \cos \theta \frac{\partial}{\partial \psi} \right), \\ \frac{\partial}{\partial q_2} &= (\sin \psi) \frac{\partial}{\partial \theta} - \frac{\cos \psi}{\sin \theta} \left(\frac{\partial}{\partial \varphi} - \cos \theta \frac{\partial}{\partial \psi} \right), \\ \frac{\partial}{\partial q_3} &= \frac{\partial}{\partial \psi},\end{aligned}$$

and we have that

$$\left[\frac{\partial}{\partial q_2}, \frac{\partial}{\partial q_1} \right] = \frac{\partial}{\partial q_3}, \quad \left[\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_3} \right] = \frac{\partial}{\partial q_2}, \quad \left[\frac{\partial}{\partial q_3}, \frac{\partial}{\partial q_2} \right] = \frac{\partial}{\partial q_1}.$$

Note that in the new coordinates the local equations defining the constraint submanifold \mathcal{M} are $\pi_4 = 0$, $\pi_5 = 0$. On the other hand, if $P : (\mathcal{T}^{TQ}TQ)|_{\mathcal{M}} = T_{\mathcal{M}}(TQ) \rightarrow \mathcal{T}^{TQ}\mathcal{M} = T\mathcal{M}$ and $Q : T_{\mathcal{M}}(TQ) \rightarrow F$ are the projectors associated with the decomposition $T_{\mathcal{M}}(TQ) = T\mathcal{M} \oplus F$, then we have that (see [11])

$$\begin{aligned}Q &= \frac{\partial}{\partial \pi_4} \otimes d\pi_4 + \frac{\partial}{\partial \pi_5} \otimes d\pi_5, \\ P &= \text{Id} - \frac{\partial}{\partial \pi_4} \otimes d\pi_4 - \frac{\partial}{\partial \pi_5} \otimes d\pi_5.\end{aligned}$$

Moreover, using that the unconstrained dynamics Γ_L is given by

$$\begin{aligned}\Gamma_L &= \dot{x} \frac{\partial}{\partial \bar{x}} + \dot{y} \frac{\partial}{\partial \bar{y}} + \dot{\theta} \frac{\partial}{\partial \theta} + \dot{\varphi} \frac{\partial}{\partial \varphi} + \dot{\psi} \frac{\partial}{\partial \psi} + \frac{k^2 \Omega}{(k^2 + r^2)} \dot{y} \frac{\partial}{\partial \pi_4} + \frac{k^2 \Omega}{(k^2 + r^2)} \dot{x} \frac{\partial}{\partial \pi_5} \\ &= \dot{x} \frac{\partial}{\partial \bar{x}} + \dot{y} \frac{\partial}{\partial \bar{y}} + \dot{q}_1 \frac{\partial}{\partial q_1} + \dot{q}_2 \frac{\partial}{\partial q_2} + \dot{q}_3 \frac{\partial}{\partial q_3} + \frac{k^2 \Omega}{(k^2 + r^2)} \dot{y} \frac{\partial}{\partial \pi_4} + \frac{k^2 \Omega}{(k^2 + r^2)} \dot{x} \frac{\partial}{\partial \pi_5},\end{aligned}$$

we deduce that the constrained dynamics is the SODE $\Gamma_{(L, \mathcal{M})}$ along \mathcal{M} defined by

$$\begin{aligned}\Gamma_{(L, \mathcal{M})} &= (P\Gamma_L|_{\mathcal{M}}) = \left(\dot{x} \frac{\partial}{\partial \bar{x}} + \dot{y} \frac{\partial}{\partial \bar{y}} + \dot{\theta} \frac{\partial}{\partial \theta} + \dot{\varphi} \frac{\partial}{\partial \varphi} + \dot{\psi} \frac{\partial}{\partial \psi} \right)|_{\mathcal{M}} \\ &= \left(\dot{x} \frac{\partial}{\partial \bar{x}} + \dot{y} \frac{\partial}{\partial \bar{y}} + \dot{q}_1 \frac{\partial}{\partial q_1} + \dot{q}_2 \frac{\partial}{\partial q_2} + \dot{q}_3 \frac{\partial}{\partial q_3} \right)|_{\mathcal{M}}.\end{aligned}\quad (7.7)$$

This implies that

$$d_{\Gamma_{(L, \mathcal{M})}}(E_L|_{\mathcal{M}}) = d_{\Gamma_{(L, \mathcal{M})}}(L|_{\mathcal{M}}) = \frac{\Omega^2 k^2}{(k^2 + r^2)}(x\dot{x} + y\dot{y})|_{\mathcal{M}}.$$

Consequently, the Lagrangian energy is a constant of the motion if and only if $\Omega = 0$.

When constructing the nonholonomic bracket on \mathcal{M} , we find that the only non-zero fundamental brackets are

$$\begin{aligned}\{x, \pi_1\}_{nh} &= r, & \{y, \pi_2\}_{nh} &= r, \\ \{q_1, \pi_2\}_{nh} &= -1, & \{q_2, \pi_1\}_{nh} &= 1, & \{q_3, \pi_3\}_{nh} &= 1, \\ \{\pi_1, \pi_2\}_{nh} &= \pi_3, & \{\pi_2, \pi_3\}_{nh} &= \frac{k^2}{(k^2 + r^2)}\pi_1 + \frac{rk^2\Omega}{(k^2 + r^2)}y, \\ \{\pi_3, \pi_1\}_{nh} &= \frac{k^2}{(k^2 + r^2)}\pi_2 - \frac{rk^2\Omega}{(k^2 + r^2)}x,\end{aligned}\quad (7.8)$$

in which the “appropriate operational” meaning has to be attached to the quasi-coordinates q_i . As a result, we have

$$\dot{f} = R_L(f) + \{f, L\}_{nh}, \text{ for } f \in C^\infty(\mathcal{M})$$

where R_L is the vector field on \mathcal{M} given by

$$R_L = \left(\frac{k^2\Omega}{(k^2 + r^2)} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) + \frac{r\Omega}{(k^2 + r^2)} \left(x \frac{\partial}{\partial q_1} + y \frac{\partial}{\partial q_2} \right) \right. \\ \left. + x(\pi_3 - k^2\Omega) \frac{\partial}{\partial \pi_1} + y(\pi_3 - k^2\Omega) \frac{\partial}{\partial \pi_2} - k^2(\pi_1 x + \pi_2 y) \frac{\partial}{\partial \pi_3} \right) |_{\mathcal{M}}.$$

Note that $R_L = 0$ if and only if $\Omega = 0$.

Now, it is clear that $Q = \mathbb{R}^2 \times SO(3)$ is the total space of a trivial principal $SO(3)$ -bundle over \mathbb{R}^2 and the bundle projection $\phi : Q \rightarrow M = \mathbb{R}^2$ is just the canonical projection on the first factor. Therefore, we may consider the corresponding Atiyah algebroid $E' = TQ/SO(3)$ over $M = \mathbb{R}^2$. Next, we describe this Lie algebroid.

Using the left-translations in $SO(3)$, one may define a diffeomorphism λ between the tangent bundle to $SO(3)$ and the product manifold $SO(3) \times \mathbb{R}^3$ (see [1]). In fact, in terms of the Euler angles, the diffeomorphism λ is given by

$$\lambda(\theta, \varphi, \psi; \dot{\theta}, \dot{\varphi}, \dot{\psi}) = (\theta, \varphi, \psi; \omega_x, \omega_y, \omega_z). \quad (7.9)$$

Under this identification between $T(SO(3))$ and $SO(3) \times \mathbb{R}^3$, the tangent action of $SO(3)$ on $T(SO(3)) \cong SO(3) \times \mathbb{R}^3$ is the trivial action

$$SO(3) \times (SO(3) \times \mathbb{R}^3) \rightarrow SO(3) \times \mathbb{R}^3, \quad (g, (h, \omega)) \mapsto (gh, \omega). \quad (7.10)$$

Thus, the Atiyah algebroid $TQ/SO(3)$ is isomorphic to the product manifold $T\mathbb{R}^2 \times \mathbb{R}^3$, and the vector bundle projection is $\tau_{\mathbb{R}^2} \circ pr_1$, where $pr_1 : T\mathbb{R}^2 \times \mathbb{R}^3 \rightarrow T\mathbb{R}^2$ and $\tau_{\mathbb{R}^2} : T\mathbb{R}^2 \rightarrow \mathbb{R}^2$ are the canonical projections.

A section of $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a pair (X, u) , where X is a vector field on \mathbb{R}^2 and $u : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a smooth map. Therefore, a global basis of sections of $T\mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is

$$e'_1 = \left(\frac{\partial}{\partial x}, 0 \right), \quad e'_2 = \left(\frac{\partial}{\partial y}, 0 \right), \\ e'_3 = (0, u_1), \quad e'_4 = (0, u_2), \quad e'_5 = (0, u_3),$$

where $u_1, u_2, u_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ are the constant maps

$$u_1(x, y) = (1, 0, 0), \quad u_2(x, y) = (0, 1, 0), \quad u_3(x, y) = (0, 0, 1).$$

In other words, there exists a one-to-one correspondence between the space $\text{Sec}(E' = TQ/SO(3))$ and the G -invariant vector fields on Q . Under this bijection, the sections e'_1 and e'_2 correspond with the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ and the sections e'_3, e'_4

and e'_5 correspond with the vertical vector fields $\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}$ and $\frac{\partial}{\partial q_3}$, respectively.

The anchor map $\rho' : E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3 \rightarrow T\mathbb{R}^2$ is the projection over the first factor and, if $[\cdot, \cdot]'$ is the Lie bracket on the space $\text{Sec}(E' = TQ/SO(3))$, then the only non-zero fundamental Lie brackets are

$$[e'_4, e'_3]' = e'_5, \quad [e'_5, e'_4]' = e'_3, \quad [e'_3, e'_5]' = e'_4.$$

From (7.9) and (7.10), it follows that the Lagrangian function $L = T$ and the constraint submanifold \mathcal{M} are $SO(3)$ -invariant. Consequently, L induces a Lagrangian function L' on $E' = TQ/SO(3)$ and, since \mathcal{M} is closed on TQ , the set of orbits $\mathcal{M}' = \mathcal{M}/SO(3)$ is a submanifold of $E' = TQ/SO(3)$ in such a way that the canonical projection $\Phi|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}' = \mathcal{M}/SO(3)$ is a surjective submersion.

Under the identification between $E' = TQ/SO(3)$ and $T\mathbb{R}^2 \times \mathbb{R}^3$, L' is given by

$$L'(x, y, \dot{x}, \dot{y}; \omega_1, \omega_2, \omega_3) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{k^2}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2),$$

where (x, y, \dot{x}, \dot{y}) and $(\omega_1, \omega_2, \omega_3)$ are the standard coordinates on $T\mathbb{R}^2$ and \mathbb{R}^3 , respectively. Moreover, the equations defining \mathcal{M}' as a submanifold of $T\mathbb{R}^2 \times \mathbb{R}^3$ are

$$\dot{x} - r\omega_2 + \Omega y = 0, \quad \dot{y} + r\omega_1 - \Omega x = 0.$$

So, we have the constrained Lagrangian system (L', \mathcal{M}') on the Atiyah algebroid $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$. Note that the constraints are not linear, and that, if Δ' is the Liouville section of the prolongation $\mathcal{T}^{E'} E'$, then the restriction to \mathcal{M}' of the vector field $(\rho')^1(\Delta')$ is not tangent to \mathcal{M}' .

Now, it is clear that the tangent bundle $TQ = T\mathbb{R}^2 \times T(SO(3)) \cong T\mathbb{R}^2 \times (SO(3) \times \mathbb{R}^3)$ is the total space of a trivial principal $SO(3)$ -bundle over $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$ and, in addition (see [43, Theorem 9.1]), the prolongation $\mathcal{T}^{E'} E'$ is isomorphic to the Atiyah algebroid associated with this principal $SO(3)$ -bundle. Therefore, the sections of the prolongation $\mathcal{T}^{E'} E' \rightarrow E'$ may be identified with the $SO(3)$ -invariant vector fields on $TQ \cong T\mathbb{R}^2 \times (SO(3) \times \mathbb{R}^3)$. Under this identification, the constrained dynamics $\Gamma_{(L', \mathcal{M}')}$ for the system (L', \mathcal{M}') is just the $SO(3)$ -invariant vector field $\Gamma_{(L, \mathcal{M})} = P(\Gamma_L|_{\mathcal{M}})$. We recall that if $\Phi : TQ \rightarrow TQ/SO(3)$ is the canonical projection, then

$$\mathcal{T}^\Phi \Phi \circ \Gamma_{(L, \mathcal{M})} = \Gamma_{(L', \mathcal{M}')} \circ \Phi. \quad (7.11)$$

Next, we give a local description of the vector field $(\rho')^1(\Gamma_{(L', \mathcal{M}')})$ on $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$ and the nonholonomic bracket $\{\cdot, \cdot\}'_{nh}$ for the constrained system (L', \mathcal{M}') . For this purpose, we consider a suitable system of local coordinates on $TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$. If we set

$$\begin{aligned} x' &= x, & y' &= y, & \pi'_3 &= k^2\omega_3, \\ \pi'_1 &= r\dot{x} + k^2\omega_2, & \pi'_2 &= r\dot{y} - k^2\omega_1, \\ \pi'_4 &= \frac{k^2}{(k^2+r^2)}(\dot{x} - r\omega_2 + \Omega y), & \pi'_5 &= \frac{k^2}{(k^2+r^2)}(\dot{y} + r\omega_1 - \Omega x), \end{aligned}$$

then $(x', y', \pi'_1, \pi'_2, \pi'_3, \pi'_4, \pi'_5)$ is a system of local coordinates on $TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$. In these coordinates the equations defining the submanifold \mathcal{M}' are $\pi'_4 = 0$ and $\pi'_5 = 0$, and the canonical projection $\Phi : TQ \rightarrow TQ/SO(3)$ is given by

$$\Phi(\bar{x}, \bar{y}, \bar{\theta}, \bar{\varphi}, \bar{\psi}; \pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (\bar{x}, \bar{y}; \pi_1, \pi_2, \pi_3, \pi_4, \pi_5). \quad (7.12)$$

Thus, from (7.7) and (7.11), it follows that

$$(\rho')^1(\Gamma_{(L', \mathcal{M}')}) = (\dot{x}' \frac{\partial}{\partial x'} + \dot{y}' \frac{\partial}{\partial y'})|_{\mathcal{M}'},$$

or, in the standard coordinates $(x, y, \dot{x}, \dot{y}; \omega_1, \omega_2, \omega_3)$ on $T\mathbb{R}^2 \times \mathbb{R}^3$,

$$\begin{aligned} (\rho')^1(\Gamma_{(L', \mathcal{M}')}) &= \left\{ \dot{x} \left(\frac{\partial}{\partial x} + \frac{\Omega k^2}{(k^2 + r^2)} \frac{\partial}{\partial \dot{y}} + \frac{\Omega r}{(k^2 + r^2)} \frac{\partial}{\partial \omega_1} \right) \right. \\ &\quad \left. + \dot{y} \left(\frac{\partial}{\partial y} - \frac{\Omega k^2}{(k^2 + r^2)} \frac{\partial}{\partial \dot{x}} + \frac{\Omega r}{(k^2 + r^2)} \frac{\partial}{\partial \omega_2} \right) \right\}|_{\mathcal{M}'}. \end{aligned}$$

On the other hand, from (7.8), (7.12) and Theorem 7.9, we deduce that the only non-zero fundamental nonholonomic brackets for the system (L', \mathcal{M}') are

$$\begin{aligned} \{x', \pi'_1\}'_{nh} &= r, & \{y', \pi'_2\}'_{nh} &= r, \\ \{\pi'_1, \pi'_2\}'_{nh} &= \pi'_3, & \{\pi'_2, \pi'_3\}'_{nh} &= \frac{k^2}{(k^2 + r^2)}\pi'_1 + \frac{rk^2\Omega}{(k^2 + r^2)}y', \\ \{\pi'_3, \pi'_1\}'_{nh} &= \frac{k^2}{(k^2 + r^2)}\pi'_2 - \frac{rk^2\Omega}{(k^2 + r^2)}x'. \end{aligned}$$

Therefore, we have that

$$f' = (\rho')^1(R_{L'})(f') + \{f', L'\}'_{nh}, \text{ for } f' \in C^\infty(\mathcal{M}'),$$

where $(\rho')^1(R_{L'})$ is the vector field on \mathcal{M}' given by

$$\begin{aligned} (\rho')^1(R_{L'}) = & \left\{ \frac{k^2\Omega}{k^2+r^2} \left(x' \frac{\partial}{\partial y'} - y' \frac{\partial}{\partial x'} \right) + \frac{r\Omega}{(k^2+r^2)} (x'(\pi'_3 - k^2\Omega) \frac{\partial}{\partial \pi'_1} \right. \\ & \left. + y'(\pi'_3 - k^2\Omega) \frac{\partial}{\partial \pi'_2} - k^2(\pi'_1 x' + \pi'_2 y') \frac{\partial}{\partial \pi'_3} \right) \Big|_{\mathcal{M}'} \right\}. \end{aligned}$$

8. CONCLUSIONS AND OUTLOOK

We have developed a geometrical description of nonholonomic mechanical systems in the context of Lie algebroids. This formalism is the natural extension of the standard treatment on the tangent bundle of the configuration space. The proposed approach also allows to deal with nonholonomic mechanical systems with symmetry, and perform the reduction procedure in a unified way. The main results obtained in the paper are summarized as follows:

- we have identified the notion of regularity of a nonholonomic mechanical system with linear constraints on a Lie algebroid, and we have characterized it in geometrical terms;
- we have obtained the constrained dynamics by projecting the unconstrained one using two different decompositions of the prolongation of the Lie algebroid along the constraint subbundle;
- we have developed a reduction procedure by stages and applied it to nonholonomic mechanical systems with symmetry. These results have allowed us to get new insights in the technique of quasicordinates;
- we have defined the operation of nonholonomic bracket to measure the evolution of observables along the solutions of the system;
- we have examined the setup of nonlinearly constrained systems;
- we have illustrated the main results of the paper in several examples.

Current and future directions of research include the in-depth study of the reduction procedure following the steps of [4, 8] for the standard case; the synthesis of so-called nonholonomic integrators [17, 21, 42] for systems evolving on Lie algebroids, and the development of a comprehensive treatment of classical field theories within the Lie algebroid formalism following the ideas by E. Martínez [54].

REFERENCES

- [1] ABRAHAM R AND MARSDEN JE, *Foundations of Mechanics*, Second edition, revised and enlarged., Benjamin/Cummings, Reading, Mass., 1978.
- [2] BATES L AND ŚNIATYCKI J, *Nonholonomic reduction*, Rep. Math. Phys. **32** (1992) 99–115.
- [3] BLOCH AM, *Nonholonomic Mechanics and Control*, Interdisciplinary Applied Mathematics Series, vol 24, Springer-Verlag, New York, 2003.
- [4] BLOCH AM, KRISHNAPRASAD PS, MARSDEN JE AND MURRAY RM, *Nonholonomic mechanical systems with symmetry*, Arch. Rational Mech. Anal. **136** (1996) 21–99.
- [5] BOJOWALD M, KOTOV A AND STROBL T, *Lie algebroid morphisms, Poisson sigma models, and off-shell closed gauge symmetries*, J. Geom. Phys. **54** (2005) 400–426.
- [6] CANNAS DA SILVA A AND WEINSTEIN A, *Geometric Models for Noncommutative Algebras*, Amer. Math. Soc., Providence, RI, 1999.
- [7] CANTRIJN F, CORTÉS J., DE LEÓN M AND MARTÍN DE DIEGO D, *On the geometry of generalized Chaplygin systems*, Math. Proc. Cambridge Philos. Soc. **132** (2002) 323–351.
- [8] CANTRIJN F, DE LEÓN M, MARRERO JC AND MARTÍN DE DIEGO D, *Reduction of nonholonomic mechanical systems with symmetries*, Rep. Math. Phys. **42** (1998) 25–45.
- [9] CANTRIJN F, DE LEÓN M, MARRERO JC AND MARTÍN DE DIEGO D, *Reduction of constrained systems with symmetries*, J. Math. Phys. **40** (1999) 795–820.
- [10] CANTRIJN F, DE LEÓN M AND MARTÍN DE DIEGO D, *On almost-Poisson structures in nonholonomic mechanics*, Nonlinearity **12** (1999) 721–737.

- [11] CANTRIJN F, DE LEÓN M, MARRERO, JC AND MARTÍN DE DIEGO D, *On almost-Poisson structures in nonholonomic mechanics II. The time-dependent framework*, Nonlinearity **13** (2000) 1379–1409.
- [12] CARIÑENA JF AND RAÑADA M, *Lagrangian systems with constraints: a geometrical approach to the method of Lagrange multipliers*, J. Phys. A: Math. Gen. **26** (1993) 1335–1351.
- [13] CARIÑENA JF, NUNES DA COSTA J.M. AND SANTOS P, *Reduction of Lie algebroid structures*, Int. J. Geom. Meth. Mod. Phys. **2** (5) (2005) 965–991.
- [14] CARIÑENA JF, NUNES DA COSTA J.M. AND SANTOS P, *Reduction of Lagrangian mechanics on Lie algebroids*, J. Geom. Phys. **57** (2007) 977–990.
- [15] CENDRA H, MARSDEN JE AND RATIU TS, *Lagrangian Reduction by Stages*, Mem. Amer. Math. Soc. **152**, no. 722, 2001.
- [16] CHAPLYGIN SA, *On the theory of the motion of nonholonomic systems. Theorem on the reducing multiplier*, Math. Sbornik **28** (2) (1911) 303–314.
- [17] CORTÉS J, *Geometric, Control and Numerical Aspects of Nonholonomic Systems*, Lect. Notes in Math., **1793**, Springer-Verlag, Berlin, 2002.
- [18] CORTÉS J AND DE LEÓN M, *Reduction and reconstruction of the dynamics of nonholonomic systems*, J. Phys. A: Math. Gen. **32** (1999) 8615–8645.
- [19] CORTÉS J, DE LEÓN M, MARRERO JC, MARTÍN DE DIEGO D AND MARTÍNEZ E, *A survey of Lagrangian mechanics and control on Lie algebroids and groupoids*, Int. J. Geom. Meth. Mod. Phys. **3** (3) (2006) 509–558.
- [20] CORTÉS J AND MARTÍNEZ E, *Mechanical control systems on Lie algebroids*, IMA J. Math. Control. Inform. **21** (2004) 457–492.
- [21] CORTÉS J AND MARTÍNEZ S, *Nonholonomic integrators*, Nonlinearity **14** (2001) 1365–1392.
- [22] CRAMPIN M, *Tangent bundle geometry for Lagrangian dynamics*, J. Phys. A: Math. Gen. **16** (1983) 3755–3772.
- [23] CUSHMAN R, KEMPPAINEN D, ŚNIATYCKI J AND BATES L, *Geometry of non-holonomic constraints*, Rep. Math. Phys. **36** (2/3) (1995) 275–286.
- [24] CUSHMAN R AND ŚNIATYCKI J, *Nonholonomic reduction for free and proper actions*, Reg. Chaotic Dyn. **7** (2002) 61–72.
- [25] EHLERS K, KOILLER J, MONTGOMERY R AND RIOS PM, *Nonholonomic systems via moving frames: Cartan equivalence and Chaplygin Hamiltonization*, In *The breadth of symplectic and Poisson geometry*. Progr. in Math., **232**, Birkhäuser, 2005, pp. 75–120.
- [26] FEDOROV YN AND JOVANOVIĆ B, *Nonholonomic LR systems as generalized Chaplygin systems with an invariant measure and flows on homogeneous spaces*, J. Nonlinear Sci. **14** (4) (2004) 341–381.
- [27] FEDOROV YN AND KOZLOV VV, *Various aspects of n -dimensional rigid body dynamics*, Amer. Math. Soc. Transl. Series 2 **168** (1995) 141–171.
- [28] FEDOROV YN AND ZENKOV D, *Discrete nonholonomic LL systems on Lie groups*, Nonlinearity **18** (5) (2005) 2211–2241.
- [29] FERNANDES RL, *Lie algebroids, holonomy and characteristic classes*, Adv. Math. **170** (1) (2002) 119–179.
- [30] GRIFONE J AND MEHDI M, *On the geometry of Lagrangian mechanics with non-holonomic constraints*, J. Geom. Phys. **30** (1999) 187–203.
- [31] HAMEL G, *Theoretische Mechanik: Eine einheitliche Einführung in die gesamte Mechanik (1949)*, Grundlehren der Mathematischen Wissenschaften, **57**, Springer-Verlag, 1978.
- [32] HIGGINS PJ AND MACKENZIE K, *Algebraic constructions in the category of Lie algebroids*, J. of Algebra **129** (1990) 194–230.
- [33] IBORT A, DE LEÓN M, MARRERO JC AND MARTÍN DE DIEGO D, *Dirac brackets in constrained dynamics*, Fortschr. Phys. **47** (1999) 459–492.
- [34] JOVANOVIĆ B, *Nonholonomic geodesic flows on Lie groups and the integrable Suslov problem on $SO(4)$* , J. Phys. A: Math. Gen. **31** (1998) 1415–1422.
- [35] JOVANOVIĆ B, *Nonholonomic left and right flows on Lie groups*, J. Phys. A: Math. Gen. **32** (1999) 8293–8302.
- [36] KLEIN J, *Espaces variationnels et mécanique*, Ann. Inst. Fourier **12** (1962) 1–124.
- [37] KOILLER J, *Reduction of some classical nonholonomic systems with symmetry*, Arch. Rational Mech. Anal. **118** (1992) 113–148.
- [38] KOON WS AND MARSDEN JE, *The Hamiltonian and Lagrangian approaches to the dynamics of nonholonomic systems*, Rep. Math. Phys. **40** (1997) 21–62.
- [39] KOON WS AND MARSDEN JE, *Poisson reduction of nonholonomic mechanical systems with symmetry*, Rep. Math. Phys. **42** (1/2) (1998) 101–134.
- [40] DE LEÓN M AND MARTÍN DE DIEGO D, *On the geometry of non-holonomic Lagrangian systems*, J. Math. Phys. **37** (1996) 3389–3414.

- [41] DE LEÓN M, MARRERO JC AND MARTÍN DE DIEGO D, *Mechanical systems with nonlinear constraints*, Int. J. Teor. Phys. **36** (4) (1997) 973–989.
- [42] DE LEÓN M, MARTÍN DE DIEGO D AND SANTAMARÍA-MERINO A, *Geometric integrators and nonholonomic mechanics*, J. Math. Phys. **45** (3) (2004) 1042–1064.
- [43] DE LEÓN M, MARRERO JC AND MARTÍNEZ E, *Lagrangian submanifolds and dynamics on Lie algebroids*, J. Phys. A: Math. Gen. **38** (2005) R241–R308.
- [44] DE LEÓN M AND RODRIGUES, PR, *Methods of Differential Geometry in Analytical Mechanics*, North Holland Math. Series **152** (Amsterdam, 1996).
- [45] LEWIS AD, *Affine connections and distributions with applications to nonholonomic mechanics*, Rep. Math. Phys. **42** (1998) 135–164.
- [46] LIBERMANN P, *Lie algebroids and mechanics*, Arch. Math. (Brno) **32** (1996) 147–162.
- [47] MARLE CHM, *Reduction of constrained mechanical systems and stability of relative equilibria*, Commun. Math. Phys. **174** (1995) 295–318.
- [48] MACKENZIE K, *Lie Groupoids and Lie Algebroids in Differential Geometry*, London Math. Soc. Lect. Note Series, **124**, Cambridge Univ. Press, Cambridge, 1987.
- [49] MARKEEV AP, *Integrability of the problem of rolling of a sphere with a multiply connected cavity filled with an ideal fluid*, Regul. Chaotic Dyn. **7** (2002) 149–151.
- [50] MARSDEN JE AND RATIU TS, *Introduction to Mechanics and Symmetry. A Basic Exposition of Classical Mechanical Systems*, Texts in Applied Mathematics, **17**, Springer-Verlag, New York, 1994.
- [51] MARTÍNEZ E, *Lagrangian mechanics on Lie algebroids*, Acta Appl. Math. **67** (2001) 295–320.
- [52] MARTÍNEZ E, MESTDAG T AND SARLET W, *Lie algebroid structures and Lagrangian systems on affine bundles*, J. Geom. Phys. **44** (1) (2002) 70–95.
- [53] MARTÍNEZ E, *Reduction in optimal control theory*, Rep. Math. Phys. **53** (2004) no. 1, 79–90.
- [54] MARTÍNEZ E, *Classical field theory on Lie algebroids: multisymplectic formalism*, preprint, arXiv:math.DG/0411352.
- [55] MESTDAG T, *Lagrangian reduction by stages for nonholonomic systems in a Lie algebroid framework*, J. Phys. A: Math. Gen. **38** (2005) 10157–10179.
- [56] MESTDAG T AND LANGEROCK B, *A Lie algebroid framework for nonholonomic systems*, J. Phys. A: Math. Gen. **38** (2005) 1097–1111.
- [57] NEIMARK J AND FUFÁEV N, *Dynamics of Nonholonomic Systems*, Translation of Mathematics Monographs, **33**, AMS, Providence, RI, 1972.
- [58] NIJENHUIS A, *Vector form brackets in Lie algebroids*, Arch. Math. (Brno) **32** (4) (1996) 317–323.
- [59] POPESCU M AND POPESCU P, *Geometric objects defined by almost Lie structures*, In *Lie Algebroids*, Banach Center Publications, **54**, 2001, pp. 217–233.
- [60] SAUNDERS D, *Prolongations of Lie groupoids and Lie algebroids*, Houston J. Math. **30** (3) (2004) 637–655.
- [61] SAUNDERS DJ, SARLET W AND CANTRIJN F, *A geometrical framework for the study of non-holonomic Lagrangian systems*, J. Phys. A: Math. Gen. **28** (1995) 3253–3268.
- [62] ŚNIATYCKI J, *Nonholonomic Noether theorem and reduction of symmetries*, Rep. Math. Phys. **42** (1998) 5–23.
- [63] SUSLOV GK, *Theoretical Mechanics*, Moskva: Gostehizda, 1946.
- [64] TAI M, *Model reduction method for nonholonomic mechanical systems with semidirect product symmetry*, In *IEEE Conference on Robotics and Automation*, New Orleans, LA, 2004, pp. 4602–4607.
- [65] VAN DER SCHAFT AJ AND MASCHKE BM, *On the Hamiltonian formulation of non-holonomic mechanical systems*, Rep. Math. Phys. **34** (1994) 225–233.
- [66] VESELOV AP AND VESELOVA LE, *Integrable nonholonomic systems on Lie groups*, Math. Notes **44** (1989) 810–819.
- [67] VINOGRADOV A. M., KUPERSHMIT B.A., *The structure of Hamiltonian mechanics*, London Math. Soc. Lect. Notes Ser., **60**, Cambridge Univ. Press, London, (1981) 173–239.
- [68] WEINSTEIN A, *Lagrangian Mechanics and groupoids*, In *Mechanics day (Waterloo, ON, 1992)*, Fields Institute Communications **7**, American Mathematical Society, 1996, pp. 207–231.
- [69] ZENKOV D, BLOCH AM, *Dynamics of the n-dimensional Suslov problem*, J. Geom. Phys. **34** (2000) no.2 121–136.
- [70] ZENKOV D, BLOCH AM, *Invariant measures of nonholonomic flows with internal degrees of freedom*, Nonlinearity **16** (2003) 1793–1807.

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